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Quasi-Poisson structures and integrable systems related to the moduli space of flat connections on a punctured Riemann sphere

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Abstract

In order to construct an integrable system on the moduli space $\mathrm{Hom}(\pi_1(S), G)/G$ of a punctured sphere S, we establish a morphism between two interesting quasi-Poisson G-manifolds, G^n and a subspace $\tilde{\mathfrak{g}}_n$ of the loop algebra of $\mathfrak{g}$. In particular, we prove a useful result about reduction in the quasi-Poisson context and we describe the construction of a quasi-Poisson structure coming from a Lie algebra splitting.

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1. Introduction

The moduli space of flat connections on a trivial principal bundle over a Riemann surface is a natural object that has been studied for a long time, in particular by Atiyah and Bott in [4]. S is a compact, connected and oriented surface which may have n marked points and whose genus is g. Let G be a Lie group whose Lie algebra $\mathfrak{g}$ admits a non-degenerate and ad-invariant symmetric bilinear form $\mathcal{B}$. The moduli space $\mathscr{M}$ is the quotient of the space $\mathscr{A}_{fl} \simeq \Omega^1(S, \mathfrak{g})$, of flat connections of the trivial bundle $S \times G$, under the action of the gauge group $\mathscr{G} \simeq C^\infty(S, G)$, of automorphisms on the bundle.

Assign to each flat connection its holonomy along a loop on S. This will give you a homomorphism $\pi_1(S) \to G$ which is well defined up to conjugacy by elements of G. There is thus a natural and well-known one-to-one correspondence between the moduli space $\mathscr{M}$ and the quotient $\mathrm{Hom}(\pi_1(S), G)/G$. Since the fundamental group $\pi_1(S)$ is the group generated by $2g + n$ elements subject to the single relation $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^{n} m_j = e$, the

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space $\mathrm{Hom}(\pi_1(S), G)/G$ is the quotient of the subset

$$\mathcal{N} := \left\{ (A_1, B_1, \ldots, M_1, \ldots) \in G^{2g+n} \,\middle|\, \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} \prod_{j=1}^{n} M_j = e \right\}$$

by the simultaneous conjugation by elements of G. This quotient is denoted by $G^{2g+n} /\!\!/ G$, so that $\mathcal{M} \simeq G^{2g+n} /\!\!/ G$. In particular, the singular variety $\mathcal{M}$ is finite-dimensional.

In 1982, Atiyah and Bott construct, in [4], a symplectic structure on the smooth part of $\mathcal{M} = \mathcal{A}_{fl}/\mathcal{G}$, in the case where $n = 0$, by reduction of the symplectic form $\omega(\varphi, \psi) = \int_S \mathcal{B} \circ (\varphi \wedge \psi)$ defined on the infinite-dimensional space of connections $\mathcal{A}$ on the trivial bundle $S \times G$.

In 1992, Fock and Rosly make, in [9,8], a finite-dimensional construction of a Poisson structure on $\mathcal{M}$ in the case where $n > 0$. They show that the symplectic leaves are obtained by fixing the conjugacy classes of the holonomies around the n marked points.

In 1994, Alekseev constructs, in [3], for arbitrary $g \geq 0$ and $n \geq 0$, a quantized algebra of functions on $\mathcal{M}$. It is generated by the entries of the monodromy matrices $M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g \in G \subset \mathbf{GL}(N)$ and subject to the quadratic relations:

$$\begin{aligned}
&R_- X_i^1 * R_-^{-1} X_i^2 = X_i^2 R_+ * X_i^1 R_+^{-1}, \\
&R_+ X_i^1 * R_+^{-1} X_j^2 = X_j^2 R_+ * X_i^1 R_+^{-1} \quad \text{if } i < j, \\
&R_+ A_i^1 * R_-^{-1} B_i^2 = B_i^2 R_+ * A_i^1 R_+^{-1},
\end{aligned}$$

where $R_\pm$ are two quantum R-matrices, $X_i^1 := X_i \otimes 1$ and $X_i^2 := 1 \otimes X_i$. In the classical limit, this non-commutative product becomes a quadratic Poisson bracket on G^{n+2g}, defined with two classical r-matrices $r_\pm$. In a tensorial formalism, it reads

$$\begin{aligned}
&\left\{X_i \overset{\otimes}{,} X_i\right\} = -r_- X_i^1 X_i^2 - X_i^2 X_i^1 r_+ + X_i^1 r_- X_i^2 + X_i^2 r_+ X_i^1, \\
&\left\{X_i \overset{\otimes}{,} X_j\right\} = -r_+ X_i^1 X_j^2 - X_j^2 X_i^1 r_+ + X_i^1 r_+ X_j^2 + X_j^2 r_+ X_i^1 \quad \text{if } i < j, \\
&\left\{A_i \overset{\otimes}{,} B_i\right\} = -r_+ A_i^1 B_i^2 - B_i^2 A_i^1 r_+ + A_i^1 r_- B_i^2 + B_i^2 r_+ A_i^1.
\end{aligned} \tag{1}$$

This Poisson structure on G^{n+2g} restricts itself to the set of G-invariant functions, and hence leads to a Poisson structure on the quotient $G^{n+2g} /\!\!/ G$. It coincides with Atiyah and Bott's in the case $n = 0$ and with Fock and Rosly's when $n > 0$.

One of the aims of Alekseev in [3] was to construct a quantum integrable system on the moduli space $\mathcal{M} = G^{n+2g} /\!\!/ G$. To do this, he introduces, for $g = 0$ and $G \subset \mathbf{GL}(N)$, the equivariant transfer map depending on a spectral parameter λ,

$$\begin{aligned}
&\mathcal{T} : G^n \longrightarrow \mathfrak{gl}(N)[\lambda] \\
&M = (M_1, \ldots, M_n) \longmapsto \mathcal{T}_M(\lambda) := (M_1 + \lambda \mathrm{Id}) \ldots (M_n + \lambda \mathrm{Id}).
\end{aligned}$$

He observes that the q-trace $F_M(\lambda) = \mathrm{tr}_q\, \mathcal{T}_M(\lambda)$ of the transfer map provides a family of commuting G-invariant elements of the quantized algebra. In the classical limit, it leads to Poisson commuting functions on $\mathcal{M}$. The calculation of independent functions thus defined shows that they form a classical integrable system when $G = \mathbf{SU}(2)$ and $g = 0$.

In this paper we will construct an integrable system on $\mathcal{M}$ when $G = \mathbf{GL}(N)$ and $g = 0$, taking Alekseev's work as a starting point. Note that there exist other constructions of integrable systems on the moduli space $\mathcal{M}$; see e.g. Goldmann [11], Jeffrey and Weitsman [12]. We are not going to talk about these works which are totally independent from the techniques that we will use here.

The choice of functions on $\mathcal{M}$ seems clear. Indeed, a large family of functions on G^n which is natural to consider consists of the pull-backs $\mathcal{T}^* F_{k,a}$ of the traces $F_{k,a}(X) := \mathrm{tr}\, X^k(a)$, $k \in \mathbb{N}$, $a \in \mathbb{C}$, where $X = X(\lambda)$ belongs to $\mathfrak{gl}(N)[\lambda]$.

What is less obvious is how to show on the one hand that these functions Poisson commute and on the other hand that they provide enough independent functions to insure integrability. To this end, it is easier to work, on the one hand on G^n, rather than $\mathscr{M} = G^n /\!\!/ G$, and on the other hand on the ambient space $\tilde{\mathfrak{g}}_n$, the subspace of $\mathfrak{gl}(N)[\lambda]$ of polynomial matrices of degree less than $n+1$. When everything is clear there, we consider the quotients under the action of G.

A list of observations and problems appears:

(a) Through the Poisson structure $\{\cdot\,,\cdot\}$ on G^n, defined by (1), the functions $F_{k,a}$ give the following Hamiltonian vector fields:

$$\mathcal{X}_{\mathcal{T}^* F_{k,a}} : \dot{\mathcal{T}}_M(\lambda) = -2k\lambda \frac{\left[\mathcal{T}_M(\lambda), \mathcal{T}_M^k(a)\right]}{\lambda - a}. \tag{2}$$

These vector fields are very similar to the well-known integrable vector fields on $\tilde{\mathfrak{g}}_n$, that were discovered independently by several authors (see e.g. [10,5]):

$$\mathcal{Y}_{k,a} : \dot{X}(\lambda) = c(a) \frac{\left[X(\lambda), X^k(a)\right]}{\lambda - a}, \tag{3}$$

where $c(a)$ depends on a only, in contrast to the coefficient in (2), which depends on λ. To use this, we gingerly alter the transfer map $\mathcal{T}$ and define

$$\begin{aligned} \mathscr{T} &: G^n \longrightarrow \tilde{\mathfrak{g}}_n \\ M = (M_1, \ldots, M_n) &\longmapsto \mathscr{T}_M(\lambda) = (\lambda M_1 + \mathrm{Id}) \ldots (\lambda M_n + \mathrm{Id}). \end{aligned}$$

Then the Hamiltonian vector fields are exactly given by the formula

$$\mathcal{X}_{\mathscr{T}^* F_{k,a}} : \dot{\mathscr{T}}_M(\lambda) = 2ka \frac{\left[\mathscr{T}_M(\lambda), \mathscr{T}_M^k(a)\right]}{\lambda - a}, \tag{4}$$

which is on the form (3).

(b) Unfortunately, neither of $\mathcal{T}$ nor $\mathscr{T}$ is a Poisson map, for any known Poisson structures on $\tilde{\mathfrak{g}}_n$. Could we find a new Poisson structure on $\tilde{\mathfrak{g}}_n$ such that the transfer map $\mathscr{T}$ becomes a Poisson morphism? A necessary condition, for such a structure, would be that the Hamiltonian vector fields coming from the functions $\operatorname{tr} X^k(a)$, $k \in \mathbb{N}$, should be $\mathcal{Y}_{k,a}$. However, the known linear structures all give $\mathcal{Y}_{k-1,a}$ (up to a constant).

(c) Beauville shows in [5] that the Hamiltonian vector fields (3) yield an integrable system on the quotient $\tilde{\mathfrak{g}}_n/G$. But, for our purpose, we need to know: is its restriction to the image of $\mathscr{T}$ in $\tilde{\mathfrak{g}}_n/G$ still an integrable system?

(d) Granted (b) and (c), do the Hamiltonian vector fields $\mathcal{X}_{\mathscr{T}^* F_{k,a}}$, $k \in \mathbb{N}$, $a \in \mathbb{C}$, form an integrable system on the moduli space?

These questions will be answered on this paper.

In the search for the Poisson structure answering the question (b), we follow Li and Parmentier's construction of quadratic Poisson structure on an associative Lie algebra [14], using R-matrices. It is the object of Section 3. Among the quadratic bivector fields thus obtained, precisely one, denoted as $\{\cdot\,,\cdot\}_1^Q$, satisfies the necessary condition demanded in (b). However it is not a Poisson structure, since the R-matrix used does not satisfy Li and Parmentier's condition. In particular, $\{\cdot\,,\cdot\}_1^Q$ is not the image of the Poisson structure $\{\cdot\,,\cdot\}$ via $\mathscr{T}$.

However, it turns out that there is a bivector field $\{\cdot\,,\cdot\}_n$ on G^n which is mapped by $\mathscr{T}$ to $\{\cdot\,,\cdot\}_1^Q$! It was constructed by Alekseev, Kosmann-Schwarzbach and Meinrenken as an example of what they call a quasi-Poisson structure (see [2]). By definition, a quasi-Poisson G-manifold is a manifold M, on which a Lie group G acts, and is equipped with a G-invariant bivector field $\{\cdot\,,\cdot\}$, satisfying, instead of the Jacobi identity, $\{f_1, \{f_2, f_3\}\} + \circlearrowleft_{1,2,3} = 2\phi_M[f_1, f_2, f_3]$, where ϕ is the Cartan 3-tensor of the Lie algebra of G. Technically less constraining, a quasi-Poisson structure offers the same possibilities as a Poisson structure: Hamiltonian vector fields, morphisms, submanifolds, ... Through the quasi-Poisson bivector field $\{\cdot\,,\cdot\}_n$, the functions $\operatorname{tr} \mathscr{T}^k(a)$ still have the vector fields $\mathcal{X}_{\mathscr{T}^* F_{k,a}}$ given by the formula (4), as Hamiltonian vector fields.

On the other side of the transfer map, we show in Section 4 that the quadratic bivector field $\{\cdot\,,\cdot\}_1^Q$, which answers question (b), surprises us by being also a quasi-Poisson structure on $\tilde{\mathfrak{g}}_n$! Hence, in particular, the morphism $\mathscr{T}$ is a quasi-Poisson morphism.

But what happens on the quotients? Quasi-Poisson structures, as Poisson structures, have a nice property for reduction: under a tangency condition, the quotient by G of a submanifold of a quasi-Poisson G-manifold inherits a (genuine!) Poisson structure. We show this result in Section 2, after a brief reminder about the quasi-Poisson manifold. Applying this to G^n and $\tilde{\mathfrak{g}}_n$, we show, in Section 5, that the quotients $G^n /\!\!/ G$ and $\mathscr{A}/G$ inherit Poisson structures, where

$$\mathscr{A} := \left\{ \mathrm{Id}\lambda^n + \lambda Y(\lambda) + \mathrm{Id} \in \tilde{\mathfrak{g}}_n \mid Y(\lambda) \in \tilde{\mathfrak{g}}_{n-2} \right\}.$$

In particular, the Poisson bivector field on $G^n /\!\!/ G$ is the same as Alekseev's in [3]. The G-invariant transfer map $\mathscr{T}$ induces hence a Poisson map $\mathscr{T}_G : G^n /\!\!/ G \mapsto \mathscr{A}/G$.

What about the integrable system? We show that Beauville's integrable system, equipped with the Poisson structure, that comes from our quasi-Poisson structure on $\tilde{\mathfrak{g}}_n$, is still an integrable system on $\mathscr{A}/G$. On the other hand, we show, in Section 6, that the Poisson map $\mathscr{T}_G$ induces a local diffeomorphism. This allows us to conclude that, as hoped, the family of functions $\mathbb{F} := (\mathrm{tr}\mathscr{T}^k(a))_{k\in\mathbb{N}, a\in\mathbb{C}}$ is an integrable system on $\mathscr{M}$.

Notice that, as a by-product of our search for an integrable system on the moduli space $\mathscr{M}$, we found a new non-trivial example of a quasi-Poisson structure, namely the quadratic bracket $\{\cdot\,,\cdot\}_1^Q$ on the loop algebra $\mathfrak{g}$. The understanding of our construction of this quasi-Poisson manifold allows us to formalize it in the more general context of a Lie algebra splitting $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$ of an associative algebra. This result is detailed in Section 7. As an example, we find an alternative construction for Adler's quadratic structure for the Toda lattice [1].

As a final note, we would like to mention that our work leads to several interesting questions. First of all, throughout this paper, we restrict ourselves to the case of $g = 0$. The question of constructing an integrable system on a surface with positive genus comes up. Secondly, we also restrict ourselves to the linear group $G = \mathbf{GL}(N, \mathbb{C})$. It allowed us to work in its Lie algebra $\mathfrak{gl}(N, \mathbb{C})$ which is an associative algebra. What could we do with other classical Lie groups G? And what about the algebraic geometry of our integrable system on the moduli space? Last but not least: do other known examples of quadratic Poisson structure fit in our general framework of quasi-Poisson manifolds?

2. Quasi-Poisson structure on G^n

This section is a short reminder about quasi-Poisson structures introduced by Alekseev, Kosmann-Schwarzbach and Meinrenken in [2]. At the end, we give an adaptation of a Poisson reduction theorem from Pedroni and Vanhaecke [15] to the case of a quasi-Poisson structure. Let us denote by $[\cdot\,,\cdot]_S$ the Schouten bracket on multivector fields. Recall that a Poisson structure on a manifold M is a bivector field $P = \{\cdot\,,\cdot\}$ on M such that $[P, P]_S = 0$, i.e. a skew-symmetric biderivation on $\mathcal{F}(M) = C^\infty(M)$ satisfying the Jacobi identity: for $f_1, f_2, f_3 \in \mathcal{F}(M)$

$$\{f_1, \{f_2, f_3\}\} + \circlearrowleft_{1,2,3} = 0, \tag{5}$$

where $\{f_1, \{f_2, f_3\}\} + \circlearrowleft_{1,2,3} = \{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\}$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We will assume throughout this text that the Lie algebra $\mathfrak{g}$ is finite-dimensional and admits a symmetric, non-degenerate, ad-invariant bilinear form $\langle\cdot|\cdot\rangle$. The non-degenerate form $\langle\cdot|\cdot\rangle$ allows us to identify $\bigwedge^k \mathfrak{g}^*$ and $\bigwedge^k \mathfrak{g}$. In particular, let us denote by ϕ the Cartan 3-form

$$\begin{aligned} \phi : \mathfrak{g}^3 &\longrightarrow \mathbb{C} \\ (x, y, z) &\longmapsto \frac{1}{2}\langle x | [y, z]\rangle \end{aligned}$$

which we usually think of as an element of $\bigwedge^3 \mathfrak{g}$: If $(e_a)_{a\in I}$ and $(\varepsilon_a)_{a\in I}$ are two dual bases of $\mathfrak{g}$ (i.e. for all $a, b \in I$, $\langle e_a | \varepsilon_b\rangle = \delta_{a,b}$), then

$$\phi = \frac{1}{12}\sum_{a,b,c\in I} \langle \varepsilon_a | [\varepsilon_b, \varepsilon_c]\rangle\, e_a \wedge e_b \wedge e_c.$$

If M is a G-manifold, then for any $x \in \mathfrak{g}$, the fundamental vector field x_M on M is defined by $x_M[f](m) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(\exp(tx)\cdot m)$, for $m \in M$ and $f \in \mathcal{F}(M)$.

More generally, the k-vector field on M associated with a k-tensor $\alpha \in \bigwedge^k \mathfrak{g}$ is denoted by α_M. A bivector field P on M is said G-invariant if $\forall x \in \mathfrak{g}$, $\mathcal{L}_{x_M} P = 0$.

Definition 1 (*[2]*). Let M be a G-manifold where G is a Lie group whose Lie algebra $\mathfrak{g}$ admits a symmetric, non-degenerate, ad-invariant bilinear form. A bivector field $P = \{\cdot\,,\cdot\}$ on M is called a *quasi-Poisson structure* on M if it is G-invariant and $[P, P]_S = \phi_M$. Then M, or (M, P), is called a *quasi-Poisson G-manifold*. This means that, instead of the Jacobi identity (5), the biderivation P satisfies, for arbitrary elements $f_1, f_2, f_3 \in \mathcal{F}(M)$,

$$\{f_1, \{f_2, f_3\}\} + \circlearrowleft_{1,2,3} = 2\phi_M[f_1, f_2, f_3].$$

A *quasi-Poisson map* between two quasi-Poisson G-manifolds $(M, \{\cdot\,,\cdot\})$ and $(M', \{\cdot\,,\cdot\}')$ is a G-equivariant map $F : M \to N$ satisfying, for $f_1, f_2 \in \mathcal{F}(M')$, $\{F^* f_1, F^* f_2\} = F^*\{f_1, f_2\}'$.

Example 2. For a G-manifold M, with G abelian, the notions of Poisson structure and quasi-Poisson structure coincide, as $\phi = 0$.

Example 3. The first non-trivial and fundamental example of a quasi-Poisson manifold is the G-manifold G where the action is given by conjugation. For $x \in \mathfrak{g}$, let us denote by $\overleftarrow{x}$ and $-\overrightarrow{x}$ (rather than x_G which we reserve for the conjugation) the fundamental vector field for the respective left actions $a \mapsto ga$ (left translation) and $a \mapsto ag^{-1}$ (right translation). For $x \in \mathfrak{g}$ the fundamental vector field for the conjugation is $x_G = \overleftarrow{x} - \overrightarrow{x}$. The bivector field

$$P_G = \frac{1}{2}\sum_a \overleftarrow{e_a} \wedge \overrightarrow{\varepsilon_a}$$

is a quasi-Poisson structure on G for the conjugation of G on itself. Indeed, using the properties of the Schouten bracket and the identities $\left[\overleftarrow{x}, \overrightarrow{y}\right] = 0$, $\left[\overleftarrow{x}, \overleftarrow{y}\right] = \overleftarrow{[x, y]}$ and $\left[\overrightarrow{x}, \overrightarrow{y}\right] = -\overrightarrow{[x, y]}$, one finds

$$\begin{aligned}
[P_G, P_G]_S &= \frac{1}{4}\sum_{ab} \overleftarrow{[e_a, e_b]} \wedge \overrightarrow{\varepsilon_a} \wedge \overrightarrow{\varepsilon_b} + \overleftarrow{e_a} \wedge \overleftarrow{e_b} \wedge \overrightarrow{[\varepsilon_a, \varepsilon_b]} \\
&= \frac{1}{4}\sum_{abc} \langle \varepsilon_a |\, [\varepsilon_b, \varepsilon_c]\rangle\, (\overleftarrow{e_c} \wedge \overrightarrow{e_a} \wedge \overrightarrow{e_b} + \overleftarrow{e_a} \wedge \overleftarrow{e_b} \wedge \overrightarrow{e_c}) \\
&= \frac{1}{12}\sum_{abc} \langle \varepsilon_a |\, [\varepsilon_b, \varepsilon_c]\rangle\, (\overleftarrow{e_a} - \overrightarrow{e_a}) \wedge (\overleftarrow{e_b} - \overrightarrow{e_b}) \wedge (\overleftarrow{e_c} - \overrightarrow{e_c}) \\
&= \phi_G
\end{aligned}$$

where we have used in the penultimate step that $\overleftarrow{\phi} - \overrightarrow{\phi} = 0$, which follows from the fact that ϕ is $\mathrm{ad}_{\mathfrak{g}}$-invariant. The same kind of formula proves that $\forall x \in \mathfrak{g}$, $[x_G, P_G]_S = 0$, such that the bivector field P_G is G-invariant. The quasi-Poisson G-manifold (G, P_G) does not depend on the choice of the basis $(e_a)_{a\in I}$. It will be referred to as the *canonical quasi-Poisson structure* on G.

This example of a quasi-Poisson G-manifold uses a canonical term $\sum_a \overleftarrow{e_a} \wedge \overrightarrow{\varepsilon_a}$ constructed with the fundamental vector fields of two different actions of G on itself. This is a particular case of a more general result due to Alekseev, Kosmann-Schwarzbach and Meinrenken, which allows us to construct a quasi-Poisson G-manifold from a quasi-Poisson $(G \times G)$-manifold ([2, theorem 5.1]). This technique, given by the following proposition, is called *fusion*.

Proposition 4 (*[2]*). *Let (M, P) be a quasi-Poisson $(G \times G)$-manifold. For $x \in \mathfrak{g}$, let us denote by x^1_M and x^2_M the fundamental vector fields associated with $x^1 = (x, 0)$ and $x^2 = (0, x)$, in $\mathfrak{g} \oplus \mathfrak{g}$, by the actions of $G \times \{e\}$ and $\{e\} \times G$ respectively. The bivector field*

$$P_{\mathrm{fus}} := P - \psi_M := P - \frac{1}{2}\sum_a e_{a^1_M} \wedge \varepsilon_{a^2_M}$$

defines on M the structure of a quasi-Poisson G-manifold with respect to the diagonal action

$$\begin{aligned}
&G \times M \to (G \times G) \times M \to M \\
&(g, m) \mapsto ((g, g), m) \mapsto (g, g) \cdot m.
\end{aligned}$$

This proposition allows us, on the one hand, to construct quasi-Poisson G-manifolds and, on the other hand, to recognize quasi-Poisson structures, when they come from fusion. We will use such an argument in Section 4.

Example 5. As suggested before, the Example 3 is a fusion: G is a $(G \times G)$-manifold for the left and the right translations. The image of the Cartan 3-tensor $\phi_2 = (\phi, \phi)$ of $\mathfrak{g} \times \mathfrak{g}$ under this action is $\overleftarrow{\phi} - \overrightarrow{\phi} = 0$. Hence $(G, P = 0)$ is a quasi-Poisson $(G \times G)$-manifold. The diagonal action is the conjugation. The fusion process gives again the quasi-Poisson G-manifold $(G, P_{\text{fus}} = \frac{1}{2} \sum_a \overleftarrow{\varepsilon_a} \wedge \overrightarrow{\varepsilon_a})$.

The fusion also allows us to define a product of quasi-Poisson G-manifolds which will be another quasi-Poisson G-manifold. If (M_1, P_1) and (M_2, P_2) are two quasi-Poisson G-manifolds, a quasi-Poisson $(G \times G)$-manifold is given by the direct product $(M_1 \times M_2, P_1 + P_2)$. In order to obtain a quasi-Poisson G-manifold let us consider its fusion product $(M_1 \times M_2, (P_1 + P_2)_{\text{fus}})$, denoted by $M_1 \circledast M_2$. The operation $\circledast$ is clearly associative.

Example 6. The fusion product of two copies of the group G, equipped with its canonical quasi-Poisson structures P_G, gives the quasi-Poisson G-manifold

$$\left(G \circledast G, \frac{1}{2} \sum_a \overleftarrow{\varepsilon_{a^1}} \wedge \overrightarrow{\varepsilon_a}^1 + \frac{1}{2} \sum_a \overleftarrow{\varepsilon_{a^2}} \wedge \overrightarrow{\varepsilon_a}^2 - \frac{1}{2} \sum_a (\overleftarrow{\varepsilon_{a^1}} - \overrightarrow{\varepsilon_a}^1) \wedge (\overleftarrow{\varepsilon_a}^2 - \overrightarrow{\varepsilon_a}^2) \right).$$

More generally, the quasi-Poisson bivector field on G^n obtained by the fusion product of n copies of (G, P_G) is

$$P_n = \frac{1}{2} \sum_{a,i} \overleftarrow{\varepsilon_a}^i \wedge \overrightarrow{\varepsilon_a}^i - \frac{1}{2} \sum_{a,i<j} (\overleftarrow{\varepsilon_a}^i - \overrightarrow{\varepsilon_a}^i) \wedge (\overleftarrow{\varepsilon_a}^j - \overrightarrow{\varepsilon_a}^j).$$

It was observed in [2] that a quasi-Poisson structure on M leads to a Poisson structure on M/G or $\mu^{-1}(e)/G$ where $\mu : M \to G$ is a G-valued moment map. The following theorem, which is inspired by Theorem 1 in [15], and which we need later on, generalizes this property. What is notable in our result is that from a quasi-Poisson G-manifold we construct on a quotient a truthful Poisson structure. It needs the following lemma:

Lemma 7 ([2]). *If (M, P) is quasi-Poisson G-manifold and G is equipped with its canonical quasi-Poisson structure P_G, the action $G \circledast M \to M$ is a quasi-Poisson map.*

Theorem 8. *Let $(M, \{\cdot, \cdot\})$ be a quasi-Poisson G-manifold and N a submanifold of M which is G-stable. Let $\mathcal{F}(M, N)^G$ be the subalgebra of functions on M which are G-invariant on N, $\mathcal{F}(N)^G$ the algebra of G-invariant functions on N and $\rho : \mathcal{F}(M, N)^G \to \mathcal{F}(N)^G$ the restriction map.*

Assume that all Hamiltonian vector fields of functions G-invariant on N are tangent to N. In other words, denoting by $\mathcal{I}(N)$ the ideal of functions on M which vanish on N, we assume that

$$\left\{ \mathcal{F}(M, N)^G, \mathcal{I}(N) \right\}_{|N} = 0$$

Then there exists a Poisson bracket, $\{\cdot, \cdot\}_{N/G}$, on $\mathcal{F}(N)^G$ such that for all $f_1, f_2 \in \mathcal{F}(M, N)^G$,

$$\{\rho f_1, \rho f_2\}_{N/G} = \rho \{f_1, f_2\}.$$

Proof. The proof that the $\{\cdot, \cdot\}_{N/G}$ is well defined is the same as in the Poisson case, using the preceding lemma (see the detailed proof in [15]). Let us prove the Jacobi identity for the new bracket $\{\cdot, \cdot\}_{N/G}$. If $f_1, f_2, f_3 \in \mathcal{F}(N)^G$ and $f_1', f_2', f_3' \in \mathcal{F}(M, N)^G$ such that $\rho f_i' = f_i$, then

$$\begin{aligned} \left\{ \{g_1, g_2\}_{N/G}, g_3 \right\}_{N/G} + \circlearrowleft_{1,2,3} &= \rho \{\{f_1, f_2\}, f_3\} + \circlearrowleft_{1,2,3} \\ &= 2\rho(\phi_M[f_1, f_2, f_3]) \\ &= 0 \end{aligned}$$

because the functions f_i are G-invariant on N and the fundamental vector fields, out of which ϕ_M is built, annihilate all G-invariant functions. □

Remark 9. When (M, P) is a quasi-Poisson G-manifold, as soon as there exists[1] an element r of $\mathfrak{g} \wedge \mathfrak{g}$ satisfying $[r, r] = -\phi$, the bivector field $P + r_M$ is a Poisson structure. This Poisson bivector field and the quasi-Poisson one

[1] Such an element r exists for any semi-simple Lie algebra.

induce, on the quotient by G of a submanifold of M, the same Poisson bracket. However, on the one hand, the quasi-Poisson bracket is more natural in the sense that it does not involve the choice of any R-matrices. On the other hand, unlike the quasi-Poisson bracket, the Poisson bracket is, in general, not G-invariant.

3. The loop algebra and its Hamiltonian structures

This section is devoted to the construction of linear and quadratic Poisson structures on the loop algebra

$$\tilde{\mathfrak{g}} := \mathfrak{g}((\lambda^{-1})) = \left\{ X = \sum_{i=-\infty}^{q} x^{[i]}\lambda^i | q \in \mathbb{Z}, x^{[i]} \in \mathfrak{g} \right\}, \tag{6}$$

where $\mathfrak{g}$ is the matrix algebra $\mathfrak{gl}(N, \mathbb{C})$. The linear brackets were first introduced by Reyman and Semenov-Tian-Shansky in [17]. For the quadratic brackets, we follow Li and Parmentier's construction [14]. In the second part of this section, we describe rigorously the tensorial formalism for the Lie algebra $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{C})$ and then for the loop algebra $\tilde{\mathfrak{g}}$. This formalism is often used in finite dimension; using it in the case of infinite-dimensional Lie algebra requires some precautions.

3.1. Construction of Poisson structures

Let us start with an associative algebra $\mathfrak{g}$ that we consider as a Lie algebra with the commutator. Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. We assume that $\mathfrak{g}$ is equipped with a symmetric, non-degenerate bilinear form $\langle\cdot|\cdot\rangle$, which satisfies, for all $x, y, z \in \mathfrak{g}$, $\langle xy|z\rangle = \langle x|yz\rangle$. It follows that $\langle\cdot|\cdot\rangle$ is ad-invariant. Denote by $\mathcal{F}(\mathfrak{g})$ the algebra of polynomial functions on $\mathfrak{g}$ generated by the linear maps $x \mapsto \langle x|y\rangle$, $y \in \mathfrak{g}$. Let us recall the construction of linear and quadratic Poisson structures on $\mathfrak{g}$. Let $R \in \mathrm{End}(\mathfrak{g})$ be a linear map on $\mathfrak{g}$ and define, for $f, g \in \mathcal{F}(\mathfrak{g})$,

$$\{f, g\}_R^L(x) := \frac{1}{2}\langle x|[R\nabla f(x), \nabla g(x)] + [\nabla f(x), R\nabla g(x)]\rangle, \tag{7}$$

$$\{f, g\}_R^Q(x) := \frac{1}{2}(\langle [x, \nabla f(x)]|R(x\nabla g(x) + \nabla g(x)x)\rangle - \langle [x, \nabla g(x)]|R(x\nabla f(x) + \nabla f(x)x)\rangle), \tag{8}$$

where $\nabla f(x)$ is defined for any $x \in \mathfrak{g}$ by

$$\forall y \in \mathfrak{g}, \quad \langle \nabla f(x)|y\rangle = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(x + ty). \tag{9}$$

A sufficient condition for R so that $\{\cdot, \cdot\}_R^L$ defines a linear Poisson structure is given by the *modified Yang–Baxter equation*

$$B_R(x, y) := [Rx, Ry] - R([Rx, y] + [x, Ry]) = -c\,[x, y],$$

where c is a constant (i.e. does not depend on x and y). For $\{\cdot, \cdot\}_R^Q$ a stronger condition is needed. According to Li and Parmentier (see [14]), if R and its skew-symmetric part (with respect to $\langle\cdot|\cdot\rangle$) both satisfy the modified Yang–Baxter equation with the same constant c, then $\{\cdot, \cdot\}_R^Q$ is a quadratic Poisson structure on $\mathfrak{g}$.

We now turn to the loop algebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$, as defined in (6). We equip $\tilde{\mathfrak{g}}$ with the Lie bracket $[x^{[i]}\lambda^i, y^{[j]}\lambda^j] = [x^{[i]}, y^{[j]}]\lambda^{i+j}$ and with the symmetric, non-degenerate and ad-invariant bilinear form, defined by

$$\langle X|Y\rangle_\sim = \sum_i \left\langle x^{[i]}|y^{[-1-i]}\right\rangle,$$

for $X = \sum_i x^{[i]}\lambda^i$ and $Y = \sum_j y^{[j]}\lambda^j$ in $\tilde{\mathfrak{g}}$. Let $\mathcal{F}(\tilde{\mathfrak{g}})$ be the algebra of functions on $\tilde{\mathfrak{g}}$ whose restriction to finite-dimensional subspaces of $\tilde{\mathfrak{g}}$ is polynomial into the linear functions $X \mapsto \langle X|Y\rangle_\sim$, $Y \in \tilde{\mathfrak{g}}$. This algebra of functions is chosen so that the biderivations that we are going to study are biderivations of $\mathcal{F}(\tilde{\mathfrak{g}})$. If $f \in \mathcal{F}(\tilde{\mathfrak{g}})$ and $X \in \tilde{\mathfrak{g}}$, the gradient $\nabla f(X)$ is defined as in formula (9) with the bilinear form $\langle\cdot|\cdot\rangle_\sim$. The loop algebra has a natural Lie algebra splitting as a vector space direct sum of two Lie subalgebras: $\tilde{\mathfrak{g}} = \mathfrak{g}[\lambda] \oplus \lambda^{-1}\mathfrak{g}[[\lambda^{-1}]]$. Let us denote by R_0 the linear map:

$$\begin{aligned} R_0 : \tilde{\mathfrak{g}} = \mathfrak{g}[\lambda] \oplus \lambda^{-1}\mathfrak{g}[[\lambda^{-1}]] &\longrightarrow \tilde{\mathfrak{g}} \\ P(\lambda) + \lambda^{-1}Q(\lambda^{-1}) &\longmapsto P(\lambda) - \lambda^{-1}Q(\lambda^{-1}) \end{aligned}$$

and for any integer l let $R_l : X(\lambda) \longmapsto R_0(\lambda^l X(\lambda))$. The assumptions used to obtain Poisson structures from the linear maps R_l are the same as in the finite-dimensional case, as is written in the following proposition:

Proposition 10. (1) [17] *The formula* (7) *defines a linear Poisson structure denoted by* $\{\cdot\,,\cdot\}_l^L$ *on the loop algebra* $\tilde{\mathfrak{g}}$ *for* $R = R_l$*, with* $l \in \mathbb{Z}$.
(2) *The formula* (8) *defines a quadratic Poisson structure denoted by* $\{\cdot\,,\cdot\}_0^Q$ *on the loop algebra* $\tilde{\mathfrak{g}}$ *for* $R = R_0$.

Proof. The result follows from

$$B_{R_l}(X, Y) = B_{R_0}(\lambda^l X(\lambda), \lambda^l Y(\lambda)) = -\left[\lambda^l X(\lambda), \lambda^l Y(\lambda)\right] = -\lambda^{2l}\,[X, Y].$$

The coefficient λ^{2l} is not a constant but we still have $[B_R(X, Y), Z] + \circlearrowleft = 0$ which directly yields the Jacobi identity for $\{\cdot\,,\cdot\}_l^L$. For the quadratic case, recall that the linear map R_0 is in addition skew-symmetric, and hence it satisfies the stronger condition needed. □

Remark 11. For $l \neq 0$, the skew-symmetric part A_l of R_l is not a solution of the modified Yang–Baxter equation. For example, for $l = 1$, we have $B_{A_1}(X, Y) = -\lambda^2\,[X, Y] + \left[x^{[-1]}, y^{[-1]}\right]$. As we will see in the next section, $\{\cdot\,,\cdot\}_1^Q$ is not a Poisson bracket.

Remark 12. The $\mathrm{ad}_{\tilde{\mathfrak{g}}}$-invariant functions are in involution with respect to both the linear and quadratic brackets $\{\cdot\,,\cdot\}_l^L$ and $\{\cdot\,,\cdot\}_l^Q$, for any integer l (i.e., if f_1 and f_2 are two $\mathrm{ad}_{\tilde{\mathfrak{g}}}$-invariant functions on $\tilde{\mathfrak{g}}$, then $\{f_1, f_2\}_l^L = \{f_1, f_2\}_l^Q = 0$). The Hamiltonian vector field generated by an $\mathrm{ad}_{\tilde{\mathfrak{g}}}$-invariant function $h \in \mathcal{F}(\tilde{\mathfrak{g}})$ is in the Lax form, for X in $\tilde{\mathfrak{g}}$,

$$\mathcal{X}_h(X) = \frac{1}{2}\,[R_l(\nabla h(X)), X]$$

for $\{\cdot\,,\cdot\}_l^L$ and

$$\mathcal{X}_h(X) = [R_l(X\nabla h(X)), X]$$

for $\{\cdot\,,\cdot\}_l^Q$. For the Poisson structures $\{\cdot\,,\cdot\}_l^L$ and $\{\cdot\,,\cdot\}_0^Q$, such Hamiltonian vector fields commute because of the Jacobi identity. We will see in Theorem 19 what happens with the quadratic bivector field $\{\cdot\,,\cdot\}_1^Q$.

For n a positive integer, let $\tilde{\mathfrak{g}}_n$ be the subspace $\tilde{\mathfrak{g}}_n := \{\sum_{i=0}^n x^{[i]}\lambda^i \,|\, x^{[i]} \in \mathfrak{g}\}$. We are going to use the transparent notation $\tilde{\mathfrak{g}}_{\geq 0} := \{\sum_{i=0}^k x^{[i]}\lambda^i \,|\, k \geq 0, x^{[i]} \in \mathfrak{g}\}$ and $\tilde{\mathfrak{g}}_{<p} := \{\sum_{i=-\infty}^{p-1} x^{[i]}\lambda^i \,|\, x^{[i]} \in \mathfrak{g}\}$ for an integer p. In [15], Pedroni and Vanhaecke have shown that the linear Poisson structures $\{\cdot\,,\cdot\}_l^L$ restrict to the subspace $\tilde{\mathfrak{g}}_n$ as long as $0 \leq l \leq n+1$. The same question arises for the quadratic bivector fields $\{\cdot\,,\cdot\}_l^Q$.

Proposition 13. *For any* $n \in \mathbb{N}^*$*, the quadratic bivector field* $\{\cdot\,,\cdot\}_l^Q$ *restricts to* $\tilde{\mathfrak{g}}_n$ *for* $l = 0$ *and* $l = 1$. *In particular,* $\{\cdot\,,\cdot\}_0^Q$ *restricts to a Poisson bracket on* $\tilde{\mathfrak{g}}_n$.

Proof. To start with, let us show that the quadratic bivector fields $\{\cdot\,,\cdot\}_l^Q$, with $l \geq 0$, restrict to $\tilde{\mathfrak{g}}_{\geq 0}$, i.e., that Hamiltonian vector fields are tangent to $\tilde{\mathfrak{g}}_{\geq 0}$. We just have to compute $\{f, g\}_l^Q(X)$ for $X \in \tilde{\mathfrak{g}}_{\geq 0}$ and f, g two linear elements of $\mathcal{F}(\tilde{\mathfrak{g}})$, where f vanishes on the subspace $\tilde{\mathfrak{g}}_{\geq 0}$. Since the gradients of such functions do not depend on X, we use the notation ∇f instead of $\nabla f(X)$. In view of definition of the bilinear form $\langle\cdot|\cdot\rangle_\sim$, we have $\nabla f \in \tilde{\mathfrak{g}}_{\geq 0}$ and thus $X\nabla f$ and $\nabla f X$ are elements of $\tilde{\mathfrak{g}}_{\geq 0}$. Then, using the definitions and properties of R_l and $\langle\cdot|\cdot\rangle_\sim$, if $l \geq 0$,

$$\begin{aligned} \{f, g\}_l^Q(X) &= -\frac{1}{2}\left(\Big\langle [X, \nabla f]\,|\,\lambda^l(X\nabla g + \nabla g X)\Big\rangle_\sim + \Big\langle [X, \nabla g]\,|\,\lambda^l(X\nabla f + \nabla f X)\Big\rangle_\sim\right) \\ &= 0. \end{aligned}$$

Hence, $\{\cdot\,,\cdot\}_l^Q$ restricts to $\tilde{\mathfrak{g}}_{\geq 0}$ for any $l \geq 0$. Now, let f, g be linear functions on $\tilde{\mathfrak{g}}_{\geq 0}$, where f vanishes on $\tilde{\mathfrak{g}}_n$ and $X \in \tilde{\mathfrak{g}}_n$. Then $\nabla f \in \tilde{\mathfrak{g}}_{<-(n+1)}$ and thus $X\nabla f, \nabla f X \in \tilde{\mathfrak{g}}_{<-1}$, and, if $l \leq 1$,

$$\begin{aligned}\{f, g\}_l^Q (X) &= \frac{1}{2}\left(\left\langle [X, \nabla f]\,|\lambda^l(X\nabla g + \nabla g X)\right\rangle_{\sim} + \left\langle [X, \nabla g]\,|\lambda^l(X\nabla f + \nabla f X)\right\rangle_{\sim}\right)\\ &= 0.\end{aligned}$$

Combining, this shows that the quadratic brackets $\{\cdot\,,\cdot\}_0^Q$ and $\{\cdot\,,\cdot\}_1^Q$ restrict to $\tilde{\mathfrak{g}}_n$. $\square$

3.2. Tensorial formalism for linear and quadratic brackets

From now on and until the end of Section 6, the Lie algebra $\mathfrak{g}$ is the matrix algebra $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{C})$, endowed with the bilinear form $\langle x|y\rangle = \operatorname{tr}(xy)$ and whose identity element is denoted by Id. The goal of this section is to use the fact that $\mathfrak{g}$ is a matrix algebra to express the brackets on $\tilde{\mathfrak{g}}$ with matrices, more exactly tensorial matrices. This formalism allows us to simplify the proof of many properties of the Poisson brackets $\{\cdot\,,\cdot\}_l^L$ and $\{\cdot\,,\cdot\}_0^Q$ and later for the non-Poisson bracket $\{\cdot\,,\cdot\}_1^Q$. We first write it for $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{C})$ and then adapt it to $\tilde{\mathfrak{g}}$. The essential point is the isomorphism between $\operatorname{End}(\mathfrak{g})$ and $\mathfrak{g} \otimes \mathfrak{g}$:

$$\begin{aligned}&\operatorname{End}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{g}\\ &R \longmapsto \sum \varepsilon_a \otimes R(e_a)\\ &(x \mapsto \langle r_1|x\rangle\, r_2) \longleftarrow r_1 \otimes r_2\end{aligned}$$

where $(e_a)_{a\in I}$ and $(\varepsilon_a)_{a\in I}$ are two dual bases of $\mathfrak{g}$ (this isomorphism does not depend on the choice of the basis). One has, if the image of R in $\mathfrak{g} \otimes \mathfrak{g}$ is $\sum_\alpha r_\alpha \otimes r'_\alpha$,

$$\langle R(x)|y\rangle = \sum_\alpha \langle r_\alpha|x\rangle \left\langle r'_\alpha|y\right\rangle.$$

Let us denote by $E_{i,j} \in \mathfrak{gl}(N, \mathbb{C})$ the matrix element with a 1 at position (i, j) and zero elsewhere, and write elements of $\mathfrak{g} \otimes \mathfrak{g}$ as matrices in $\mathfrak{gl}(N^2, \mathbb{C})$ whose entry at position $N(i-1)+k, N(j-1)+l$ is given by

$$(x \otimes y)_{i,j,k,l} := x_{i,j}\, y_{k,l}.$$

The tensor product $\mathfrak{g} \otimes \mathfrak{g}$ inherits, from the associative algebra of $\mathfrak{gl}(N^2, \mathbb{C})$, the structure of a Lie algebra and two $\mathfrak{g}$-valued trace maps: $\operatorname{tr}_1(x \otimes y) := (\operatorname{tr} x)y$ and $\operatorname{tr}_2(x \otimes y) := (\operatorname{tr} y)x$.

The linear bracket (7) for a linear map R whose image in $\mathfrak{g} \otimes \mathfrak{g}$ is denoted by $r = \sum_\alpha r_\alpha \otimes r'_\alpha$ reads, for any $f, g \in \mathcal{F}(\mathfrak{g})$ and $x \in \mathfrak{g}$,

$$\begin{aligned}\{f, g\}_R^L (x) &= \frac{1}{2}\langle x|[R\nabla f(x), \nabla g(x)] + [\nabla f(x), R\nabla g(x)]\rangle\\ &= \frac{1}{2}\langle [\nabla g(x), x]\,|R\nabla f(x)\rangle + \frac{1}{2}\langle [x, \nabla f(x)]\,|R\nabla g(x)\rangle\\ &= \frac{1}{2}\sum_\alpha \left(\langle r_\alpha|\nabla f(x)\rangle \left\langle r'_\alpha|\,[\nabla g(x), x]\right\rangle + \langle r_\alpha|\nabla g(x)\rangle \left\langle r'_\alpha|\,[x, \nabla f(x)]\right\rangle\right).\end{aligned}$$

For $x_{ij} := \left\langle E_{ij}|\,\cdot\right\rangle$ a coordinate function on $\mathfrak{g}$, one has $\nabla x_{ij}(x) = E_{ji}$ which leads, using the definition of $\langle\cdot|\cdot\rangle$, to

$$\begin{aligned}\left\{x_{ij}, x_{kl}\right\}_R^L (x) &= \frac{1}{2}\sum_\alpha ((r_\alpha)_{i,j}(x r'_\alpha - r'_\alpha x)_{k,l} + (r_\alpha)_{k,l}(r'_\alpha x - x r'_\alpha)_{i,j})\\ &= \frac{1}{2}\sum_\alpha \left(r_\alpha \otimes \left[x, r'_\alpha\right] + \left[r'_\alpha, x\right] \otimes r_\alpha\right)_{i,j,k,l}\\ &= \frac{1}{2}\left([(\mathrm{Id} \otimes x), r] - \left[(x \otimes \mathrm{Id}), r^*\right]\right)_{i,j,k,l},\end{aligned}$$

where $r^* = \sum_\alpha r'_\alpha \otimes r_\alpha$ is the image in $\mathfrak{g}\otimes\mathfrak{g}$ of the linear adjoint map R^*. All the information of the Poisson bracket is encoded in the $\mathcal{F}(\mathfrak{g})$-valued matrix $\left\{x \overset{\otimes}{,} x\right\}^L_R$ defined by

$$\left\{x \overset{\otimes}{,} x\right\}^L_R := \sum_{i,j,k,l} \left\{x_{ij}, x_{kl}\right\}^L_R E_{ij} \otimes E_{kl}$$

that we simply write as

$$\left\{x \overset{\otimes}{,} x\right\}^L_R = \frac{1}{2}\left([\mathrm{Id}\otimes x, r] - \left[x\otimes \mathrm{Id}, r^*\right]\right). \tag{10}$$

In the same way, the tensorial matrix for the quadratic bivector field

$$\{f,g\}^Q_R(x) = \frac{1}{2}(\langle [x,\nabla f(x)] | R(x\nabla g(x) + \nabla g(x) x)\rangle - \langle [x,\nabla g(x)] | R(x\nabla f(x) + \nabla f(x) x)\rangle)$$

reads

$$\left\{x \overset{\otimes}{,} x\right\}^Q_R = \left[x\otimes x, \frac{r-r^*}{2}\right] + (\mathrm{Id}\otimes x)\frac{r+r^*}{2}(x\otimes \mathrm{Id}) - (x\otimes \mathrm{Id})\frac{r+r^*}{2}(\mathrm{Id}\otimes x). \tag{11}$$

The computation is left to the reader.

When the linear map R is moreover skew-symmetric, i.e. r^* is equal to $-r$, then this formula simplifies to $\left\{x \overset{\otimes}{,} x\right\}^Q_R = [x\otimes x, r]$.

We now pass to the case of the loop algebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g} = \mathfrak{gl}(N)$. A technical difficulty arises considering the tensor square of $\tilde{\mathfrak{g}}$. Let us explain our convention used to denote elements of $\tilde{\mathfrak{g}}\otimes\tilde{\mathfrak{g}}$. It will be of practical use to write $x\otimes y\lambda^p\mu^q \in \mathfrak{g}\otimes\mathfrak{g}[\![\lambda,\lambda^{-1},\mu,\mu^{-1}]\!]$ instead of the more correct notation $(x\lambda^p)\otimes(y\lambda^q)$. We also need to enlarge the usual tensor product $\tilde{\mathfrak{g}}\otimes\tilde{\mathfrak{g}}$ by defining the space $\mathcal{T}_2(\tilde{\mathfrak{g}})$:

$$\mathcal{T}_2(\tilde{\mathfrak{g}}) := \left\{ \sum_{-l\le p+q\le l} \alpha_{p,q}\lambda^p\mu^q | \alpha_{p,q} \in \mathfrak{g}\otimes\mathfrak{g}, l\in\mathbb{Z}\right\}.$$

Similarly, one defines $\mathcal{T}_3(\tilde{\mathfrak{g}})$:

$$\mathcal{T}_3(\tilde{\mathfrak{g}}) := \left\{ \sum_{-l\le p+q+r\le l} \alpha_{p,q,r}\lambda^p\mu^q\nu^r | \alpha_{p,q,r} \in \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}, l\in\mathbb{Z}\right\}.$$

The bilinear form $\langle\cdot|\cdot\rangle_\sim$ of $\tilde{\mathfrak{g}}$ leads to a pairing $\langle\cdot|\cdot\rangle_\otimes$:

$$\begin{aligned}
&\langle\cdot|\cdot\rangle_\otimes : \mathcal{T}_2(\tilde{\mathfrak{g}}) \times (\tilde{\mathfrak{g}}\otimes\tilde{\mathfrak{g}}) \to \mathbb{C}\\
&(x\otimes y\lambda^p\mu^q, z\otimes t\lambda^r\mu^s) \mapsto \left\langle x\lambda^p|z\lambda^r\right\rangle_\sim \left\langle y\lambda^q|t\lambda^s\right\rangle_\sim .
\end{aligned}$$

We also define the adjoint on $\mathcal{T}_2(\tilde{\mathfrak{g}})$ and $\tilde{\mathfrak{g}}\otimes\tilde{\mathfrak{g}}$ by the formula

$$(x\otimes y\lambda^p\mu^q)^* = y\otimes x\lambda^q\mu^p.$$

Then, for $A\in\mathcal{T}_2(\tilde{\mathfrak{g}})$ and $B\in\tilde{\mathfrak{g}}\otimes\tilde{\mathfrak{g}}$: $\langle A^*|B^*\rangle_\otimes = \langle A|B\rangle_\otimes$.

If $(e_a)_{a\in I}$ and $(\varepsilon_a)_{a\in I}$ are dual bases of $\mathfrak{g}$, then dual bases of $\tilde{\mathfrak{g}}$ are given by $(e_a\lambda^p)_{a\in I,p\in\mathbb{Z}}$ and $(\varepsilon_a\lambda^{-p-1})_{a\in I,p\in\mathbb{Z}}$. Let $\mathcal{E}(\tilde{\mathfrak{g}})$ be the subspace of $\mathrm{End}(\tilde{\mathfrak{g}})$

$$\mathcal{E}(\tilde{\mathfrak{g}}) := \{R\in\mathrm{End}(\tilde{\mathfrak{g}}) | \exists l\in\mathbb{Z}, \forall p\in\mathbb{Z}, R(\mathfrak{g}\lambda^p)\subset\tilde{\mathfrak{g}}_{p-l,p+l}\}.$$

As for $\mathfrak{g} = \mathfrak{gl}(N,\mathbb{C})$, there exists an injective linear map between $\mathcal{E}(\tilde{\mathfrak{g}})$ and $\mathcal{T}_2(\tilde{\mathfrak{g}})$, denoted by β:

$$\begin{aligned}
&\beta : \mathcal{E}(\tilde{\mathfrak{g}}) \longrightarrow \mathcal{T}_2(\tilde{\mathfrak{g}})\\
&R \longmapsto \sum_{p\in\mathbb{Z},a\in I} \varepsilon_a\lambda^{-p-1}\otimes R(e_a\mu^p).
\end{aligned} \tag{12}$$

For $R \in \mathcal{E}(\tilde{\mathfrak{g}})$, this map satisfies the two following properties:

$$\forall X, Y \in \tilde{\mathfrak{g}}, \quad \langle R(X(\lambda))|Y(\lambda)\rangle_{\sim} = \langle \beta(R)|X(\lambda) \otimes Y(\mu)\rangle_{\otimes}$$
$$\beta(R)^* = \beta(R^*).$$

In particular, $r_l = \beta(R_l)$ in $\mathcal{T}_2(\tilde{\mathfrak{g}})$ is given by

$$r_l = \sum_{\substack{p\in\mathbb{Z} \\ i,j\in[\![1,N]\!]}} \eta_{p+l} E_{ji}\lambda^{-p-1} \otimes E_{ij}\mu^{p+l}, \tag{13}$$

where the coefficient η_p is equal to $+1$ if $p \geq 0$ and -1 if $p < 0$. The matrix which encodes the Poisson structure is a matrix in λ and μ:

$$\left\{X(\lambda) \overset{\otimes}{,} X(\mu)\right\} = \sum_{\substack{i,j,k,l\in[\![1,N]\!] \\ p,q\in\mathbb{Z}}} \left\{{x_{ij}}^{[p]}, {x_{kl}}^{[q]}\right\} \lambda^p \mu^q E_{ij} \otimes E_{kl}.$$

Let us introduce $\mathfrak{t}_0 = \sum_{i,j} E_{ij} \otimes E_{ji} \in \mathfrak{g} \otimes \mathfrak{g}$. One computes easily $(\lambda - \mu)r_l = 2\lambda^l \mathfrak{t}_0$ and the techniques used for (10) yield, in this case,

$$\left\{X(\lambda) \overset{\otimes}{,} X(\mu)\right\}_l^L = \frac{1}{\lambda-\mu}\left(\lambda^l[\mathrm{Id} \otimes X(\mu), \mathfrak{t}_0] + \mu^l[X(\lambda) \otimes \mathrm{Id}, \mathfrak{t}_0]\right).$$

This formula needs to be understood as follows: for any $X \in \tilde{\mathfrak{g}}$, the polynomial in $\lambda, \lambda^{-1}, \mu, \mu^{-1}$, $(\lambda^l[\mathrm{Id}\otimes X(\mu), \mathfrak{t}_0]+ \mu^l[X(\lambda) \otimes \mathrm{Id}, \mathfrak{t}_0])$ is divisible by $\lambda - \mu$. For the quadratic brackets, we have, as in (11),

$$\begin{aligned}\left\{X(\lambda) \overset{\otimes}{,} X(\mu)\right\}_l^Q = {} & \frac{\lambda^l + \mu^l}{\lambda-\mu}[X(\lambda) \otimes X(\mu), \mathfrak{t}_0] + \frac{\lambda^l - \mu^l}{\lambda-\mu}\,((\mathrm{Id} \otimes X(\mu))\mathfrak{t}_0(X(\lambda) \otimes \mathrm{Id}) \\ & - (X(\lambda) \otimes \mathrm{Id})\mathfrak{t}_0(\mathrm{Id} \otimes X(\mu)))\,. \end{aligned} \tag{14}$$

In particular

$$\left\{X(\lambda) \overset{\otimes}{,} X(\mu)\right\}_0^Q = \frac{2}{\lambda-\mu}[X(\lambda) \otimes X(\mu), \mathfrak{t}_0].$$

With this formalism, the skew-symmetry of the bracket $\{\cdot\,,\cdot\}$ reads

$$\left\{X(\lambda) \overset{\otimes}{,} X(\mu)\right\} = -\mathfrak{t}_0 \left\{X(\mu) \overset{\otimes}{,} X(\lambda)\right\} \mathfrak{t}_0,$$

Moreover, the Leibniz rule of $\{\cdot\,,\cdot\}$ is written, for any $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$, as

$$\left\{A(\lambda) \overset{\otimes}{,} B(\mu)C(\mu)\right\} = \mathrm{Id} \otimes B(\mu) \left\{A(\lambda) \overset{\otimes}{,} C(\mu)\right\} + \left\{A(\lambda) \overset{\otimes}{,} B(\mu)\right\} \mathrm{Id} \otimes C(\mu).$$

As a first use of this formalism, let us look at the following proposition. Recall that G is a Lie group integrating the Lie algebra $\mathfrak{g}$.

Proposition 14. *The adjoint action of the Lie group G on the loop algebra $\tilde{\mathfrak{g}}$ defines, for any $g \in G$, a Poisson map $\rho_g : X(\lambda) \mapsto gX(\lambda)g^{-1}$ on $\tilde{\mathfrak{g}}$ for the structures $\{\cdot\,,\cdot\}_l^L$, $l \in \mathbb{Z}$ and $\{\cdot\,,\cdot\}_0^Q$.*

Proof. Let us compute the bracket $\left\{x_{ij} \circ \rho_g, x_{kl} \circ \rho_g\right\}_l^L (X)$. It is given by the entry at position i, j, k, l of the tensorial matrix $\left\{gX(\lambda)g^{-1} \overset{\otimes}{,} gX(\mu)g^{-1}\right\}_l^L$. Using the Leibniz rule, with the constant matrices g and g^{-1},

$$\begin{aligned}\left\{gX(\lambda)g^{-1} \overset{\otimes}{,} gX(\mu)g^{-1}\right\}_l^L &= g \otimes g \left\{X(\lambda) \overset{\otimes}{,} X(\mu)\right\}_l^L g^{-1} \otimes g^{-1} \\ &= \frac{1}{\lambda-\mu} g \otimes g(\lambda^l\,[\mathrm{Id} \otimes X(\mu), \mathfrak{t}_0] + \mu^l\,[X(\lambda) \otimes \mathrm{Id}, \mathfrak{t}_0])g^{-1} \otimes g^{-1} \\ &= \frac{1}{\lambda-\mu}\left(\lambda^l\left[\mathrm{Id} \otimes gX(\mu)g^{-1}, \mathfrak{t}_0\right] + \mu^l\left[gX(\lambda)g^{-1} \otimes \mathrm{Id}, \mathfrak{t}_0\right]\right)\end{aligned}$$

We obtain therefore

$$\left\{x_{ij}\circ\rho_g, x_{kl}\circ\rho_g\right\}_l^L(X) = \left\{x_{ij}, x_{kl}\right\}_l^L(gXg^{-1}), \quad \forall i,j,k,l \in [\![1,N]\!].$$

Thus we have $\left\{x_{ij}\circ\rho_g, x_{kl}\circ\rho_g\right\}_l^L = \left\{x_{ij}, x_{kl}\right\}_l^L \circ \rho_g$. This argument works for the quadratic bivector fields $\{\cdot\,,\cdot\}_l^Q$ as well. □

4. Quasi-Poisson bracket on the loop algebra

We have shown in the previous section that two quadratic structures on the loop algebra $\tilde{\mathfrak{g}}$ are restricted to the subspaces $\tilde{\mathfrak{g}}_n$, namely $\{\cdot\,,\cdot\}_0^Q$ and $\{\cdot\,,\cdot\}_1^Q$. $\{\cdot\,,\cdot\}_0^Q$ is a Poisson structure by Proposition 10, whereas $\{\cdot\,,\cdot\}_1^Q$ is not. The aim of this section is to study this structure $\{\cdot\,,\cdot\}_1^Q$ on $\tilde{\mathfrak{g}}$, in particular to show that it is a quasi-Poisson bracket for the conjugation by $G = \mathbf{GL}(N,\mathbb{C})$ on $\tilde{\mathfrak{g}}$:

$$\begin{aligned}\rho : G\times\tilde{\mathfrak{g}} &\to \tilde{\mathfrak{g}}\\ (g,X) &\mapsto gXg^{-1}.\end{aligned}$$

Thus, as suggested before, $\{\cdot\,,\cdot\}_1^Q$ will not be a Poisson bracket on $\tilde{\mathfrak{g}}$, but it will lead to a Poisson bracket on the quotient $\tilde{\mathfrak{g}}_n/G$.

For $x \in \mathfrak{g}$, let us denote by $\underline{x}$ the fundamental vector field on $\tilde{\mathfrak{g}}$ for the conjugation. It is given by: $\underline{x} = \overleftarrow{x} - \overrightarrow{x}$, where $\overleftarrow{x}$ and $-\overrightarrow{x}$ stand for the fundamental vector fields of the left and the right translations respectively. For $x \in \tilde{\mathfrak{g}}$, $\overleftarrow{x}$ and $-\overrightarrow{x}$ also designate the two infinitesimal actions of $\tilde{\mathfrak{g}}$ on itself[2]: $\forall x \in \tilde{\mathfrak{g}}$, $\forall X \in \tilde{\mathfrak{g}}$, $\overleftarrow{x}(X) = xX$ and $\overrightarrow{x}(X) = Xx$. We have to show that $\left[\{\cdot\,,\cdot\}_1^Q, \{\cdot\,,\cdot\}_1^Q\right]_S = \underline{\phi}$, where ϕ is the Cartan 3-tensor of $\mathfrak{g}$. For that purpose, we are going to use the fusion procedure detailed in Proposition 4. If A and S are the skew-symmetric and symmetric parts of R_1, $\{\cdot\,,\cdot\}_1^Q$ is also written as

$$\{f,g\}_1^Q(X) = \{f,g\}_a(X) + \{f,g\}_s(X),$$

where

$$\begin{aligned}\{f,g\}_a(X) &= \langle A(\nabla f(X)X)|\nabla g(X)X\rangle_\sim - \langle A(X\nabla f(X))|X\nabla g(X)\rangle_\sim\\ \{f,g\}_s(X) &= \langle S(X\nabla f(X))|\nabla g(X)X\rangle_\sim - \langle S(\nabla f(X)X)|X\nabla g(X)\rangle_\sim .\end{aligned}$$

The symmetric and skew-symmetric parts A and S are given for $x\lambda^k \in \tilde{\mathfrak{g}}$ by

$$A(x\lambda^k) = \left|\begin{array}{ll} x\lambda^{k+1} & \text{if } k+1>0\\ -x\lambda^{k+1} & \text{if } k+1<0\\ 0 & \text{if } k=-1\end{array}\right. \quad\text{and}\quad S(x\lambda^k) = x\delta_{k,-1}. \tag{15}$$

The idea is to show that $\{\cdot\,,\cdot\}_a$ is a quasi-Poisson bracket on the $(G\times G)$-space $\tilde{\mathfrak{g}}$ and to see then that $\{\cdot\,,\cdot\}_1^Q$ is the quasi-Poisson bracket on the G-space $\tilde{\mathfrak{g}}$, which is the fusion of $\{\cdot\,,\cdot\}_a$. Consider the left $(G\times G)$-action on the loop algebra $\tilde{\mathfrak{g}}$:

$$\begin{aligned}G\times G\times\tilde{\mathfrak{g}} &\to \tilde{\mathfrak{g}}\\ (g_1,g_2,X) &\mapsto g_1Xg_2^{-1}.\end{aligned}$$

The Lie algebra of the product $G\times G$ is the direct sum $\mathfrak{g}\oplus\mathfrak{g}$. The fundamental vector field on $\tilde{\mathfrak{g}}$ corresponding to the action of $G\times G$ is given, for $x^1+y^2 \in \mathfrak{g}\oplus\mathfrak{g}$, by $\widehat{x^1+y^2} = \overleftarrow{x} - \overrightarrow{y}$. We denote by ϕ_2 the Cartan 3-tensor of the Lie algebra $\mathfrak{g}\oplus\mathfrak{g}$: $\phi_2 = (\phi,\phi)$. We have to show that $\{\cdot\,,\cdot\}_a$ is a $(G\times G)$-invariant bivector field and that $\left[\{\cdot\,,\cdot\}_a, \{\cdot\,,\cdot\}_a\right]_S = \widehat{\phi_2}$.

[2] Even if we often use capital letters to designate elements of $\tilde{\mathfrak{g}}$, we choose the lower case here in order to preserve homogeneity for infinitesimal actions.

Let $a := \beta(A)$ be the image of A in $\mathcal{T}_2(\tilde{\mathfrak{g}})$ via the map β defined by the formula (12). Then $\{\cdot\,,\cdot\}_a = \frac{1}{2}(\overleftarrow{a} - \overrightarrow{a})$. Explicitly, we have, in view of (13) or (15),

$$a = \sum_{\substack{1\le i,j\le N\\ p>0}} E_{ji}\lambda^{-p} \wedge E_{ij}\mu^p = \sum_{\substack{1\le i,j\le N\\ p>0}} E_{ji}\lambda^{-p} \otimes E_{ij}\mu^p - \sum_{\substack{1\le i,j\le N\\ p>0}} E_{ij}\lambda^{p} \otimes E_{ji}\mu^{-p}.$$

A short computation gives quickly the following equalities:

Lemma 15. *The 2-tensor a satisfies, $\forall x \in \mathfrak{g}$, $[a, x] = 0$ and $\frac{1}{4}[a, a] = \phi - \phi_\sim$, where $\phi_\sim \in \mathcal{T}_3(\tilde{\mathfrak{g}})$ denotes the Cartan 3-tensor of the Lie algebra $\tilde{\mathfrak{g}}$ equipped with the ad-invariant bilinear form $\langle x\lambda^p | y\lambda^q\rangle'_\sim = \delta_{p,-q}\operatorname{tr}(xy)$.*

Proof. Let $x \in \mathfrak{g}$ and $p \in \mathbb{N}^*$. The bracket $\left[\sum_{1\le i,j\le n} E_{ji}\lambda^{-p} \wedge E_{ij}\mu^p, x\right]$ reads

$$\begin{aligned}\sum_{1\le i,j\le n} \left[E_{ji}\lambda^{-p} \wedge E_{ij}\mu^p, x\right] &= \sum_{1\le i,j\le n} \left[E_{ji}, x\right]\lambda^{-p} \wedge E_{ij}\mu^p + \sum_{1\le i,j\le n} E_{ji}\lambda^{-p} \wedge \left[E_{ij}, x\right]\mu^p\\
&= \sum_{1\le i,j,k,l\le n} \big\langle E_{kl} \big| \left[E_{ji}, x\right]\big\rangle E_{lk}\lambda^{-p} \wedge E_{ij}\mu^p\\
&\quad + \sum_{1\le i,j,k,l\le n} \big\langle E_{kl} \big| \left[E_{ij}, x\right]\big\rangle E_{ji}\lambda^{-p} \wedge E_{lk}\mu^p\\
&= 0.\end{aligned}$$

Hence, $[a, x] = \sum_{p>0}\left[\sum_{1\le i,j\le n} E_{ji}\lambda^{-p} \wedge E_{ij}\mu^p, x\right] = 0$. Denoting by $(e_a)_{a\in I} = (E_{ij}\lambda^p)_{\substack{1\le i,j\le N\\ p>0}}$ the basis of $\tilde{\mathfrak{g}}_{>0}$, $(\varepsilon_a)_{a\in I} = (E_{ji}\lambda^p)_{\substack{1\le i,j\le N\\ p<0}}$ its dual basis in $\tilde{\mathfrak{g}}_{<0}$ with respect to the bilinear form $\langle\cdot|\cdot\rangle'_\sim$ and $(h_c)_{c\in J}$ an orthogonal basis of $\mathfrak{g}$, one develops the bracket $[a, a]$ on $\bigwedge^3 \tilde{\mathfrak{g}}$ through

$$\begin{aligned}[a, a] &= \sum_{a,b\in I} \varepsilon_a \wedge [e_a, \varepsilon_b] \wedge e_b + [\varepsilon_a, \varepsilon_b] \wedge e_a \wedge e_b - \varepsilon_b \wedge [\varepsilon_a, e_b] \wedge e_a + \varepsilon_a \wedge \varepsilon_b \wedge [e_a, e_b]\\
&= -\sum_{a,b,c\in I} + \langle e_c | [\varepsilon_a, \varepsilon_b]\rangle'_\sim\, \varepsilon_c \wedge e_a \wedge e_b + \langle \varepsilon_c | [e_a, e_b]\rangle'_\sim\, e_c \wedge \varepsilon_a \wedge \varepsilon_b\\
&\quad -2 \sum_{a,b\in I,c\in J} \langle h_c | [e_a, \varepsilon_b]\rangle'_\sim\, h_c \wedge \varepsilon_a \wedge e_b.\end{aligned}$$

In addition, the Cartan 3-tensors $\phi_\sim$ of $\tilde{\mathfrak{g}}$ and ϕ of $\mathfrak{g}$ read

$$\begin{aligned}\phi_\sim &= \frac{1}{4}\sum_{a,b,c} \langle e_a | [\varepsilon_b, \varepsilon_c]\rangle'_\sim\, \varepsilon_a \wedge e_b \wedge e_c + \langle \varepsilon_a | [e_b, e_c]\rangle'_\sim\, e_a \wedge \varepsilon_b \wedge \varepsilon_c\\
&\quad + \frac{1}{2}\sum_{a,b\in I,c\in J} \langle \varepsilon_a | [e_b, h_c]\rangle'_\sim\, e_a \wedge \varepsilon_b \wedge h_c + \frac{1}{12}\sum_{a,b,c\in J} \langle h_a | [h_b, h_c]\rangle'_\sim\, h_a \wedge h_b \wedge h_c\end{aligned}$$

$$\phi = \frac{1}{12}\sum_{a,b,c\in J} \langle h_a | [h_b, h_c]\rangle\, h_a \wedge h_b \wedge h_c = \frac{1}{12}\sum_{a,b,c\in J} \langle h_a | [h_b, h_c]\rangle'_\sim\, h_a \wedge h_b \wedge h_c.$$

Hence we have the expected formula: $\frac{1}{4}[a, a] = \phi - \phi_\sim$. $\square$

Remark 16. Note that the bilinear form that appears naturally in Lemma 15 is $\langle\cdot|\cdot\rangle'_\sim$, rather than $\langle\cdot|\cdot\rangle_\sim$, which was used to define the brackets $\{\cdot\,,\cdot\}_1^Q$ and $\{\cdot\,,\cdot\}_a$ and the 2-tensor a. Actually, for proving that $\tilde{\mathfrak{g}}$ is a quasi-Poisson $(G \times G)$-manifold and G-manifold respectively, the only bilinear form which plays a role is the one on the Lie algebra $\mathfrak{g}$ of the Lie group G. Indeed we see in the following proposition that, when we consider the bivector field $\frac{1}{4}(\overleftarrow{a} - \overrightarrow{a})$, we just need the fact that $\phi_\sim$ is an $\operatorname{ad}_{\tilde{\mathfrak{g}}}$-invariant 3-tensor.

Proposition 17. *The bivector field $\{\cdot\,,\cdot\}_a$ is a quasi-Poisson structure on the $(G \times G)$-space $\tilde{\mathfrak{g}}$.*

Proof. To show that $\{\cdot\,,\cdot\}_a$ is $(G\times G)$-invariant, we just need to compute its Lie derivative with respect to fundamental vector fields. Let $x^1 + y^2 = (x, y) \in \mathfrak{g}\oplus\mathfrak{g}$. We have $\mathcal{L}_{\widehat{x^1+y^2}}\{\cdot\,,\cdot\}_a = \frac{1}{2}\left[\overleftarrow{a} - \overrightarrow{a}, \overleftarrow{x} - \overrightarrow{y}\right]_S = \overleftarrow{[a, x]} - \overrightarrow{[a, y]} = 0$.

Now let us compute the Schouten bracket: $\frac{1}{4}\left(\left[\overleftarrow{a}-\overrightarrow{a}, \overleftarrow{a}-\overrightarrow{a}\right]_S\right)=\frac{1}{4}(\overleftarrow{[a,a]}-\overrightarrow{[a,a]})=\overleftarrow{\phi}-\overrightarrow{\phi}-\overleftarrow{\phi}_\sim+\overrightarrow{\phi}_\sim$. Since $\phi_\sim$ is $\mathrm{ad}_{\tilde{\mathfrak{g}}}$-invariant, we have $\overleftarrow{\phi}_\sim-\overrightarrow{\phi}_\sim=0$ and thus $\frac{1}{4}\left(\left[\overleftarrow{a}-\overrightarrow{a}, \overleftarrow{a}-\overrightarrow{a}\right]_S\right)=\widehat{\phi_2}$. □

Before stating the theorem, let us interpose the following lemma which gives a major property of the bivector field $\{\cdot\,,\cdot\}_1^Q$. We thank Camille Laurent for this observation.

Lemma 18. *Let $(M,\{\cdot\,,\cdot\})$ be a quasi-Poisson G-manifold. Let $f_1, f_2\in\mathcal{F}(M)$ be two functions on M, such that*
(1) $f_1\in\mathcal{F}(M)^G$,
(2) $\{f_1,f_2\}=0$.
Then the Hamiltonian vector fields $\mathcal{X}_{f_1}$ and $\mathcal{X}_{f_2}$ commute:

$$\left[\mathcal{X}_{f_1},\mathcal{X}_{f_2}\right]=0.$$

Proof. We just need the graded Jacobi identity of the Schouten bracket and the definition of a quasi-Poisson bivector field $\{\cdot\,,\cdot\}=\pi$.

$$\begin{aligned}\left[\mathcal{X}_{f_1},\mathcal{X}_{f_2}\right]&=\left[[\pi,f_1]_S\,,[\pi,f_2]_S\right]_S\\&=\frac{1}{2}\left[\left[[\pi,\pi]_S\,,f_1\right]_S,f_2\right]_S+\left[\pi,\left[[\pi,f_1]_S\,,f_2\right]_S\right]_S\\&=\frac{1}{2}\left[\left[\underline{\phi_{\mathfrak{g}}},f_1\right]_S,f_2\right]_S+[\pi,\{f_1,f_2\}]_S\\&=0.\quad\square\end{aligned}$$

Using Proposition 4, we obtain the announced result:

Theorem 19. *The bivector field $\{\cdot\,,\cdot\}_1^Q$ is a quasi-Poisson structure on the G-space $\tilde{\mathfrak{g}}$. For this bracket, the Hamiltonian vectors fields of $\mathrm{ad}_{\tilde{\mathfrak{g}}}$-invariant functions commute.*

Proof. The bivector field obtained on $\tilde{\mathfrak{g}}$ by the fusion of the $(G\times G)$-action is $\{\cdot\,,\cdot\}_a-\widehat{\psi}$, where $\psi=\frac{1}{2}\sum E_{ji}^1\wedge E_{ij}^2$.

$$\widehat{\psi}=\frac{1}{2}\sum\widehat{E_{ji}^1}\wedge\widehat{E_{ij}^2}=-\frac{1}{2}\sum\overleftarrow{E_{ji}}\wedge\overrightarrow{E_{ij}}=-\{\cdot\,,\cdot\}_s\,.$$

Thus the fusion of the $(G\times G)$-action gives the bivector field $\{\cdot\,,\cdot\}_a+\{\cdot\,,\cdot\}_s=\{\cdot\,,\cdot\}_1^Q$.

For the second part of the theorem, recall that we have seen in Section 3 that the $\mathrm{ad}_{\tilde{\mathfrak{g}}}$-invariant functions are in involution for the quadratic brackets $\{\cdot\,,\cdot\}_l^Q$, $l\in\mathbb{Z}$. Lemma 18 allows us to conclude. □

Remark 20. It is possible to show directly that $\{\cdot\,,\cdot\}_1^Q$ is a quasi-Poisson bracket, i.e., without using the fusion Proposition and calculation of the Schouten bracket. It suffices to compute the Jacobiator with the decomposition $\{\cdot\,,\cdot\}_1^Q=\{\cdot\,,\cdot\}_a+\{\cdot\,,\cdot\}_s$. We will not do this computation here, but just outline how to do this. For f_1, f_2, f_3, three linear functions on $\tilde{\mathfrak{g}}$, and X in $\tilde{\mathfrak{g}}$, denote by L_1, L_2, L_3 their gradients at X. One computes

$$\begin{aligned}\nabla\{f_2,f_3\}_a(X)&=A(L_2X)L_3-A(L_3X)L_2+L_2A(XL_3)-L_3A(XL_2),\\\nabla\{f_2,f_3\}_s(X)&=S(XL_2)L_3-S(XL_3)L_2+L_2S(L_3X)-L_3S(L_2X).\end{aligned}$$

One should simply calculate the four double brackets $\left\{f_1,\{f_2,f_3\}_{a,s}\right\}_{a,s}(X)$, in order to write

$$\begin{aligned}\left\{f_1,\{f_2,f_3\}_1^Q\right\}_1^Q(X)+\circlearrowleft_{1,2,3}&=\left\{f_1,\{f_2,f_3\}_a\right\}_a(X)+\circlearrowleft_{1,2,3}+\left\{f_1,\{f_2,f_3\}_s\right\}_s(X)+\circlearrowleft_{1,2,3}\\&\quad+\left\{f_1,\{f_2,f_3\}_s\right\}_a(X)+\circlearrowleft_{1,2,3}+\left\{f_1,\{f_2,f_3\}_a\right\}_s(X)+\circlearrowleft_{1,2,3}\\&=\langle L_1X|B_A(L_2X,L_3X)\rangle_\sim-\langle XL_1|B_A(XL_2,XL_3)\rangle_\sim\\&\quad-\left\langle XL_1|B_A(L_2X,L_3X)-\frac{1}{2}B'_{R_1}(L_2X,L_3X)+\frac{1}{2}B'_{R_1}(L_3X,L_2X)\right\rangle_\sim+\circlearrowleft_{1,2,3}\\&\quad+\left\langle L_1X|B_A(XL_2,XL_3)-\frac{1}{2}B'_{R_1}(XL_2,XL_3)+\frac{1}{2}B'_{R_1}(XL_3,XL_2)\right\rangle_\sim+\circlearrowleft_{1,2,3},\end{aligned}$$

where, for $Y, Z \in \tilde{\mathfrak{g}}$,

$$B_{R_1}(Y,Z) = [R_1Y, R_1Z] - R_1([R_1Y, Z] + [Y, R_1Z]),$$
$$B'_{R_1}(Y,Z) = R_1^*[R_1Y, Z] - R_1^*\left[Y, R_1^*Z\right] - \left[R_1Y, R_1^*Z\right].$$

The computation of B_A and B'_{R_1} gives

$$\begin{aligned}\left\{f_1, \{f_2, f_3\}_1^Q\right\}_1^Q (X) + \circlearrowleft_{1,2,3} &= \left\langle (L_1X)^{[-1]} | \left[(L_2X)^{[-1]}, (L_3X)^{[-1]}\right]\right\rangle \\ &\quad - \left\langle (XL_1)^{[-1]} | \left[(XL_2)^{[-1]}, (XL_3)^{[-1]}\right]\right\rangle - \left\langle (XL_1)^{[-1]} | \left[(L_2X)^{[-1]}, (L_3X)^{[-1]}\right]\right\rangle + \circlearrowleft_{1,2,3} \\ &\quad + \left\langle (L_1X)^{[-1]} | \left[(XL_2)^{[-1]}, (XL_3)^{[-1]}\right]\right\rangle + \circlearrowleft_{1,2,3} \\ &= 2\underline{\phi}[f_1, f_2, f_3](X).\end{aligned}$$

Remark 21. We have seen in Proposition 13 that the bivector field $\{\cdot\,,\cdot\}_1^Q$ is restricted to the sub-G-space $\tilde{\mathfrak{g}}_n$ for each positive integer n. Hence $\tilde{\mathfrak{g}}_n$ equipped with $\{\cdot\,,\cdot\}_1^Q$ is a quasi-Poisson G-manifold.

5. Two examples of quasi-Poisson reduction

In this section, we apply Theorem 8 to the two main examples of the quasi-Poisson G-manifold that we have constructed: (G^n, P_n) in Section 2 and $(\tilde{\mathfrak{g}}_n, \{\cdot\,,\cdot\}_1^Q)$ in Section 4.

5.1. Poisson structure on a quotient of G^n

We fix some integer $n \in \mathbb{N}$ and we use the shorthand G for the linear group $\mathbf{GL}(N, \mathbb{C})$. Consider in G^n the subset

$$\mathcal{N} := \left\{(M_1, \dots, M_n) \in G^n | M_1 \dots M_n = \mathrm{Id}\right\}.$$

$\mathcal{N}$ is clearly a G-stable submanifold of G^n with respect to $g \cdot (M_1, \dots, M_n) = (gM_1g^{-1}, \dots, gM_ng^{-1})$. By Example 6, G^n equipped with the biderivation

$$\{\cdot\,,\cdot\}_n = P_n = \frac{1}{2}\sum_{a,i} \overleftarrow{\varepsilon_a}^i \wedge \overrightarrow{\varepsilon_a}^i - \frac{1}{2}\sum_{a,i<j} (\overleftarrow{\varepsilon_a}^i - \overrightarrow{\varepsilon_a}^i) \wedge (\overleftarrow{\varepsilon_a}^j - \overrightarrow{\varepsilon_a}^j)$$

is a quasi-Poisson G-manifold.

In [2], the authors define the notion of G-valued moment map for a quasi-Poisson G-manifold. Then they prove that the quasi-Poisson structure induces a Poisson structure on quotients by a Hamiltonian reduction. Thus, they show that the quotient $G^n /\!\!/ G$ inherits a Poisson structure from $\{\cdot\,,\cdot\}_n$. In addition, they claim that this Poisson structure is the same as what can be found in [4,8,3]. Here, we are going to use a simpler argument to prove that $\{\cdot\,,\cdot\}_n$ induces a Poisson structure on $G^n /\!\!/ G$. It is an example of the use of Theorem 8 and of the tensorial formalism.

By Theorem 8, to yield a Poisson structure on the quotient $\mathcal{N}/G$, we just have to show that the Hamiltonian vector fields associated with functions on G^n, which are G-invariant on $\mathcal{N}$, are tangent to $\mathcal{N}$. Let us check this point.

Since G is reductive, the algebra $\mathcal{F}(G^n, \mathcal{N})^G$ of polynomial functions on G^n which are G-invariant on $\mathcal{N}$ is generated by the algebra $\mathcal{F}(G^n)^G$ of G-invariant functions on G^n and the ideal $\mathcal{I}(\mathcal{N})$ of functions on G^n which vanish on the submanifold $\mathcal{N}$ (see for example the chapter Reductive groups in [18]). In addition, according to Procesi [16, Theorem 1.3],

$$\mathcal{F}(G^n)^G = \langle \mathrm{tr}(M_{\alpha_1} \dots M_{\alpha_p}) | p \in \mathbb{N}, (\alpha_1, \dots, \alpha_p) \in [\![1, n]\!]^p\rangle$$

and, by definition of $\mathcal{N}$,

$$\mathcal{I}(\mathcal{N}) = \langle (M_1 \dots M_n - \mathrm{Id})_{kl} | k, l \in [\![1, N]\!]\rangle.$$

Thus, in Theorem 8, the assumption $\left\{\mathcal{F}(G^n, \mathcal{N})^G, \mathcal{I}(\mathcal{N})\right\}_{|\mathcal{N}} = 0$ is satisfied if and only if

$$\left.\begin{aligned}\left\{M_1 \dots M_n \overset{\otimes}{,} M_1 \dots M_n\right\}_n = 0 \\ \left\{\mathrm{tr}(M_{\alpha_1} \dots M_{\alpha_p}), M_1 \dots M_n\right\}_n = 0\end{aligned}\right. \quad \text{in } \mathcal{N}$$

for all $p \in \mathbb{N}$, $(\alpha_1, \dots, \alpha_p) \in [\![1, n]\!]^p$. Using the tensorial formalism, we get

$$\begin{aligned}\left\{M_i \overset{\otimes}{,} M_i\right\}_n &= -(M_i \otimes \mathrm{Id})\mathfrak{t}_0(\mathrm{Id} \otimes M_i) + (\mathrm{Id} \otimes M_i)\mathfrak{t}_0(M_i \otimes \mathrm{Id}),\\ \left\{M_i \overset{\otimes}{,} M_j\right\}_n &= (M_i \otimes \mathrm{Id})\mathfrak{t}_0(\mathrm{Id} \otimes M_j) + (\mathrm{Id} \otimes M_j)\mathfrak{t}_0(M_i \otimes \mathrm{Id})\\ &\quad - \mathfrak{t}_0(M_i \otimes M_j) - (M_i \otimes M_j)\mathfrak{t}_0 \qquad \text{if } i < j,\end{aligned} \tag{16}$$

where $\mathfrak{t}_0$ is again defined by $\mathfrak{t}_0 = \sum_{k,l \in \mathbb{Z}} E_{k,l} \otimes E_{l,k}$. The linearity of tr gives

$$\left\{\mathrm{tr}(M_{\alpha_1} \dots M_{\alpha_p}), M_1 \dots M_n\right\}_n = \mathrm{tr}_1 \left\{M_{\alpha_1} \dots M_{\alpha_p} \overset{\otimes}{,} M_1 \dots M_n\right\}_n .$$

On the other hand, the Leibniz rule allows us to write, using the notation $A_i := M_{\alpha_1} \dots M_{\alpha_i}$, $B_i := M_{\alpha_i} \dots M_{\alpha_p}$, $S_j := M_1 \dots M_j$ and $T_j := M_j \dots M_n$,

$$\left\{M_{\alpha_1} \dots M_{\alpha_p} \overset{\otimes}{,} M_1 \dots M_n\right\}_n = \sum_{\substack{i \in [\![1,p]\!]\\ j \in [\![1,n]\!]}} (A_{i-1} \otimes S_{j-1}) \left\{M_{\alpha_i} \overset{\otimes}{,} M_j\right\}_n (B_{i+1} \otimes T_{j+1}).$$

Combining this with formulas (16), one first obtains

$$\begin{aligned}\left\{M_{\alpha_1} \dots M_{\alpha_p} \overset{\otimes}{,} M_1 \dots M_n\right\}_n = \sum_{i=1}^{p} \sum_{j=1}^{\alpha_i - 1} &\big((A_{i-1} \otimes S_{j-1})\mathfrak{t}_0(B_i \otimes T_j) + (A_i \otimes S_j)\mathfrak{t}_0(B_{i+1} \otimes T_{j+1})\\ &- (A_{i-1} \otimes S_j)\mathfrak{t}_0(B_i \otimes T_{j+1}) - (A_i \otimes S_{j-1})\mathfrak{t}_0(B_{i+1} \otimes T_j)\big)\\ &+ \big((A_{i-1} \otimes S_{\alpha_i})\mathfrak{t}_0(B_i \otimes T_{\alpha_i+1}) - (A_i \otimes S_{\alpha_i-1})\mathfrak{t}_0(B_{i+1} \otimes T_{\alpha_i})\big)\\ &+ \sum_{j=\alpha_i+1}^{n} \big((-A_{i-1} \otimes S_{j-1})\mathfrak{t}_0(B_i \otimes T_j) - (A_i \otimes S_j)\mathfrak{t}_0(B_{i+1} \otimes T_{j+1})\\ &+ (A_{i-1} \otimes S_j)\mathfrak{t}_0(B_i \otimes T_{j+1}) + (A_i \otimes S_{j-1})\mathfrak{t}_0(B_{i+1} \otimes T_j)\big).\end{aligned}$$

Changing the index j with a shift, a great number of terms in these sums disappear. Recall also that, by definition of T_k and S_k, and using the fact that we are on $\mathscr{N}$, we have $S_0 = T_1 = S_n = T_{n+1} = \mathrm{Id}$. Hence

$$\begin{aligned}&\left\{M_{\alpha_1} \dots M_{\alpha_p} \overset{\otimes}{,} M_1 \dots M_n\right\}_n\\ &\quad = 2\sum_{i=1}^{p} ((A_{i-1} \otimes \mathrm{Id})\mathfrak{t}_0(B_i \otimes \mathrm{Id}) - (A_i \otimes \mathrm{Id})\mathfrak{t}_0(B_{i+1} \otimes \mathrm{Id}))\\ &\qquad + \sum_{i=1}^{p} \big((A_i \otimes S_{\alpha_i})\mathfrak{t}_0(B_{i+1} \otimes T_{\alpha_i+1}) - (A_{i-1} \otimes S_{\alpha_i+1})\mathfrak{t}_0(B_i \otimes T_{\alpha_i})\big)\\ &\quad = 2\big(\mathfrak{t}_0(B_1 \otimes \mathrm{Id}) - (A_p \otimes \mathrm{Id})\mathfrak{t}_0\big) + \sum_{i=1}^{p} \big((A_{i-1} \otimes S_{\alpha_i-1})(M_{\alpha_i} \otimes M_{\alpha_i}\mathfrak{t}_0 - \mathfrak{t}_0 M_{\alpha_i} \otimes M_{\alpha_i})(B_{i+1} \otimes T_{\alpha_i+1})\big).\\ &\quad = 2\left[\mathfrak{t}_0, M_{\alpha_1} \dots M_{\alpha_p} \otimes \mathrm{Id}\right].\end{aligned}$$

Taking tr_1 and $M_{\alpha_1} \dots M_{\alpha_p} = M_1 \dots M_n$ on $\mathscr{N}$ gives the expected result: Hamiltonian vector fields associated with functions G-invariant on $\mathscr{N}$ are tangent to the sub-G-manifold $\mathscr{N}$ so that we can use Theorem 8: the quotient $G^n /\!/ G := \mathscr{N}/G$ inherits a Poisson bracket. This structure is the same as those constructed in [4] and [8].

5.2. *Poisson structure on a quotient of* $\tilde{\mathfrak{g}}_n$

We propose here a second example of reduction with Theorem 8. This quotient will play an important role in the construction of an integrable system on the moduli space $G^n /\!/ G$.

Let $\mathscr{A}$ be the affine G-invariant subspace

$$\mathscr{A} := \left\{ X = \mathrm{Id}\lambda^n + \sum_{i=1}^{n-1} x^{[i]}\lambda^i + \mathrm{Id} | x^{[i]} \in \mathfrak{g} \right\} \subset \tilde{\mathfrak{g}}_n. \tag{17}$$

Since G is reductive, the algebra $\mathcal{F}(\tilde{\mathfrak{g}}_n, \mathscr{A})^G$ of polynomial functions on $\tilde{\mathfrak{g}}_n$ which are G-invariant on $\mathscr{A}$ is generated by the algebra

$$\mathcal{F}(\tilde{\mathfrak{g}}_n)^G = \langle \operatorname{tr}(x^{[\alpha_1]} \dots x^{[\alpha_p]}) | p \in \mathbb{N}, (\alpha_1, \dots, \alpha_p) \in [\![0, n]\!]^p \rangle$$

of G-invariant functions on $\tilde{\mathfrak{g}}_n$, and the ideal

$$\mathcal{I}(\mathscr{A}) = \langle x_{ij}^{[0]} - \delta_{ij}, x_{ij}^{[n]} - \delta_{ij} | i, j \in [\![1, N]\!] \rangle,$$

of null-functions on $\mathscr{A}$. Thus we just need to show the following equalities on $\mathscr{A}$, for any $p \in \mathbb{N}$ and $(\alpha_1, \dots, \alpha_p) \in [\![0, n]\!]^p$:

$$\left\{\operatorname{tr}(x^{[\alpha_1]} \dots x^{[\alpha_p]}), x^{[0]}\right\}_1^Q = 0, \qquad \left\{x^{[0]} \overset{\otimes}{,} x^{[0]}\right\}_1^Q = 0, \qquad \left\{x^{[n]} \overset{\otimes}{,} x^{[0]}\right\}_1^Q = 0,$$
$$\left\{\operatorname{tr}(x^{[\alpha_1]} \dots x^{[\alpha_p]}), x^{[n]}\right\}_1^Q = 0, \qquad \left\{x^{[0]} \overset{\otimes}{,} x^{[n]}\right\}_1^Q = 0, \qquad \left\{x^{[n]} \overset{\otimes}{,} x^{[n]}\right\}_1^Q = 0.$$

Recall from (14) the tensorial form of the quasi-Poisson bracket $\{\cdot\,, \cdot\}_1^Q$:

$$\begin{aligned}\left\{X(\lambda) \overset{\otimes}{,} X(\mu)\right\}_1^Q = {} & \frac{\lambda + \mu}{\lambda - \mu}[X(\lambda) \otimes X(\mu), \mathrm{t}_0] + (\mathrm{Id} \otimes X(\mu))\mathrm{t}_0(X(\lambda) \otimes \mathrm{Id}) \\ & - (X(\lambda) \otimes \mathrm{Id})\mathrm{t}_0(\mathrm{Id} \otimes X(\mu)),\end{aligned}$$

and the equality in $\tilde{\mathfrak{g}}$: $(A \otimes B)\mathrm{t}_0 = \mathrm{t}_0(B \otimes A)$. This implies, taking $\mu = 0$ in the previous expression, for some X in $\mathscr{A}$,

$$\begin{aligned}\left\{X(\lambda) \overset{\otimes}{,} x^{[0]}\right\}_1^Q &= \left\{X(\lambda) \overset{\otimes}{,} X(0)\right\}_1^Q \\ &= [X(\lambda) \otimes x^{[0]}, \mathrm{t}_0] + (\mathrm{Id} \otimes x^{[0]})\mathrm{t}_0(X(\lambda) \otimes \mathrm{Id}) - (X(\lambda) \otimes \mathrm{Id})\mathrm{t}_0(\mathrm{Id} \otimes x^{[0]}) \\ &= 0.\end{aligned}$$

In particular, for any integer $\alpha \in [\![0, n]\!]$, we have $\left\{x^{[\alpha]} \overset{\otimes}{,} x^{[0]}\right\}_1^Q = 0$ on $\mathscr{A}$. In the same way, we compute the tensorial bracket $\left\{X(\lambda) \overset{\otimes}{,} x^{[n]}\right\}_1^Q$ by taking a limit as μ tends to ∞, for some X in $\mathscr{A}$:

$$\begin{aligned}\left\{X(\lambda) \overset{\otimes}{,} x^{[n]}\right\}_1^Q &= \frac{1}{\mu^n}\left\{X(\lambda) \overset{\otimes}{,} \lim_{\mu \to \infty} X(\mu)\right\}_1^Q \\ &= -[X(\lambda) \otimes x^{[n]}, \mathrm{t}_0] + (\mathrm{Id} \otimes x^{[n]})\mathrm{t}_0(X(\lambda) \otimes \mathrm{Id}) - (X(\lambda) \otimes \mathrm{Id})\mathrm{t}_0(\mathrm{Id} \otimes x^{[n]}) \\ &= -2\left[X(\lambda) \otimes \mathrm{Id}, \mathrm{t}_0\right].\end{aligned}$$

Thus $\left\{x^{[0]} \overset{\otimes}{,} x^{[n]}\right\}_1^Q = \left\{x^{[n]} \overset{\otimes}{,} x^{[n]}\right\}_1^Q = 0$ on $\mathscr{A}$ and

$$\begin{aligned}\left\{\operatorname{tr}(x^{[\alpha_1]} \dots x^{[\alpha_p]}), x^{[n]}\right\}_1^Q{}_{|\mathscr{A}} &= -2\sum_{i=1}^{p} \operatorname{tr}_1((x^{[\alpha_1]} \dots x^{[\alpha_{i-1}]} \otimes \mathrm{Id})\left[x^{[\alpha_i]} \otimes \mathrm{Id}, \mathrm{t}_0\right](x^{[\alpha_{i+1}]} \dots x^{[\alpha_p]} \otimes \mathrm{Id}))_{|\mathscr{A}} \\ &= -2\sum_{i=1}^{p}(x^{[\alpha_{i+1}]} \dots x^{[\alpha_p]} x^{[\alpha_1]} \dots x^{[\alpha_i]} - x^{[\alpha_i]} \dots x^{[\alpha_p]} x^{[\alpha_1]} \dots x^{[\alpha_{i-1}]})_{|\mathscr{A}} \\ &= 0.\end{aligned}$$

Hence $\left\{\mathcal{F}(\tilde{\mathfrak{g}}_n, \mathscr{A})^G, \mathcal{I}(\mathscr{A})\right\}_1^Q{}_{|\mathscr{A}} = 0$. We have thus shown the following proposition:

Proposition 22. *The quotient $\mathscr{A}/G$ inherits a Poisson structure from the quadratic quasi-Poisson bracket $\{\cdot\,, \cdot\}_1^Q$ defined on $\tilde{\mathfrak{g}}_n$.*

6. An integrable system on the moduli space

Before constructing an integrable system on $\mathscr{A}/G$ and $\mathscr{M}$, it is advisable to specify what we mean by an integrable system. In this paper, we consider integrability in the sense of Liouville. The definition is the following one:

Definition 23. Let $(M, \{\cdot\,,\cdot\})$ be a Poisson manifold, with algebra of functions $\mathcal{F}(M)$. Let $2r$ be the maximal rank of the Poisson structure on M. A subalgebra $\mathbb{F} \in \mathcal{F}(M)$ is called an *integrable system* on M if it satisfies

(1) $\mathbb{F}$ is involutive (i.e., $\forall f_1, f_2 \in \mathbb{F}, \{f_1, f_2\} = 0$);
(2) $\mathbb{F}$ is generated by $\dim M - r$ independent functions on M.

The goal of this section is to construct an integrable system on the regular part of the moduli space $\mathscr{M} = G^n /\!\!/ G$. To this end, we use the equivariant transfer map

$$\begin{aligned} \mathscr{T} : G^n &\longrightarrow \tilde{\mathfrak{g}}_n \\ (M_1, \ldots, M_n) &\longmapsto (\lambda M_1 + \mathrm{Id}) \ldots (\lambda M_n + \mathrm{Id}), \end{aligned}$$

which induces a transfer map on the quotients: $\mathscr{T}_G : G^n /\!\!/ G \mapsto \mathscr{A}/G$, where $\mathscr{A}$ was defined in (17). Let us still denote by $\{\cdot\,,\cdot\}_n$ and $\{\cdot\,,\cdot\}_1^Q$ the Poisson structures obtained by reduction in Section 5 on $G^n /\!\!/ G$ and $\mathscr{A}/G$ respectively. When X belongs to $\mathscr{A}$, the determinant $\det(y\mathrm{Id} - X(\lambda))$ is a polynomial in λ and y. Let $\mathbb{F}$ be the subalgebra of functions on $\mathscr{A}/G$ thus defined:

$$\mathbb{F} := \langle \operatorname{tr} X^k(a) \mid k \in \mathbb{N}, a \in \mathbb{C} \rangle = \left\langle \varphi_{p,q} \mid \det(y\mathrm{Id} - X(\lambda)) = \sum_{p,q} \varphi_{p,q}(X) \lambda^p y^q \right\rangle.$$

We will show that $\mathbb{F}$ is an integrable system on $\mathscr{A}/G$ and that $\mathscr{T}_G^*\mathbb{F}$ is an integrable system on the moduli space $G^n /\!\!/ G$. We will do this by first showing that the functions are in involution and then computing the number of independent functions.

6.1. An involutive family of functions

First of all, $\mathbb{F}$ is a subalgebra of $\mathrm{ad}_{\tilde{\mathfrak{g}}}$-invariant functions on $\tilde{\mathfrak{g}}_n$. Hence, according to the Remark 12 and considering the construction of the Poisson structure $\{\cdot\,,\cdot\}_1^Q$ on $\mathscr{A}/G$ (Proposition 22), $\mathbb{F}$ is an involutive subalgebra of $\mathcal{F}(\mathscr{A}/G)$. In order to deduce that $\mathscr{T}^*\mathbb{F}$ is involutive in $\mathcal{F}(G^n /\!\!/ G)$, we prove the following proposition:

Proposition 24. *The transfer map $\mathscr{T}$ is a morphism of quasi-Poisson G-manifolds between $(G^n, \{\cdot\,,\cdot\}_n)$ and $(\tilde{\mathfrak{g}}_n, \{\cdot\,,\cdot\}_1^Q)$. It induces then a Poisson morphism $\mathscr{T}_G : G^n /\!\!/ G \mapsto \mathscr{A}/G$.*

Proof. The result can be shown computing the image of $\{\cdot\,,\cdot\}_n$ by the transfer map $\mathscr{T}$, and recognizing that it is precisely $\{\cdot\,,\cdot\}_1^Q$ (see [13]). However, a less tiresome calculation in the tensorial formalism, inspired by Alekseev's computation in [3], gives immediately the result. Recall from (16) that the bracket $\{\cdot\,,\cdot\}_n$ is given by

$$\begin{aligned} \left\{M_i \overset{\otimes}{,} M_i\right\}_n &= -(M_i \otimes \mathrm{Id})\mathfrak{t}_0(\mathrm{Id} \otimes M_i) + (\mathrm{Id} \otimes M_i)\mathfrak{t}_0(M_i \otimes \mathrm{Id}), \\ \left\{M_i \overset{\otimes}{,} M_j\right\}_n &= (M_i \otimes \mathrm{Id})\mathfrak{t}_0(\mathrm{Id} \otimes M_j) + (\mathrm{Id} \otimes M_j)\mathfrak{t}_0(M_i \otimes \mathrm{Id}) - \mathfrak{t}_0(M_i \otimes M_j) - (M_i \otimes M_j)\mathfrak{t}_0 \quad \text{if } i < j. \end{aligned}$$

Observe that, denoting by $M_i(\lambda)$ the factor $\lambda M_i + \mathrm{Id}$, the tensorial polynomial brackets $\left\{M_i(\lambda) \overset{\otimes}{,} M_j(\mu)\right\}_n$ satisfy

$$\begin{aligned} (\lambda - \mu)\left\{M_i(\lambda) \overset{\otimes}{,} M_i(\mu)\right\}_n &= (\lambda + \mu)[M_i(\lambda) \otimes M_i(\mu), \mathfrak{t}_0] + (\lambda - \mu)((\mathrm{Id} \otimes M_i(\mu))\mathfrak{t}_0(M_i(\lambda) \otimes \mathrm{Id}) \\ &\quad - (M_i(\lambda) \otimes \mathrm{Id})\mathfrak{t}_0(\mathrm{Id} \otimes M_i(\mu))), \\ \left\{M_i(\lambda) \overset{\otimes}{,} M_j(\mu)\right\}_n &= (M_i(\lambda) \otimes \mathrm{Id})\mathfrak{t}_0(\mathrm{Id} \otimes M_j(\mu)) - \mathfrak{t}_0(M_i(\lambda) \otimes M_j(\mu)) \\ &\quad + (\mathrm{Id} \otimes M_j(\mu))\mathfrak{t}_0(M_i(\lambda) \otimes \mathrm{Id}) - (M_i(\lambda) \otimes M_j(\mu))\mathfrak{t}_0 \quad \text{if } i < j, \end{aligned}$$

so that, using the Leibniz identity, the bracket of the product $\mathscr{T}(\lambda)$ satisfies

$$\begin{aligned} \left\{\mathscr{T}(\lambda) \overset{\otimes}{,} \mathscr{T}(\mu)\right\}_n &= \frac{\lambda + \mu}{\lambda - \mu}[\mathscr{T}(\lambda) \otimes \mathscr{T}(\mu), \mathfrak{t}_0] \\ &\quad + (\mathrm{Id} \otimes \mathscr{T}(\mu))\mathfrak{t}_0(\mathscr{T}(\lambda) \otimes \mathrm{Id}) - (\mathscr{T}(\lambda) \otimes \mathrm{Id})\mathfrak{t}_0(\mathrm{Id} \otimes \mathscr{T}(\mu)). \end{aligned}$$

One recognizes the bracket $\{\cdot\,,\cdot\}_1^Q$ of $\tilde{\mathfrak{g}}_n$, given by the formula (14). □

Since the subalgebra $\mathbb{F}$ is involutive on $(\mathscr{A}/G, \{\cdot\,,\cdot\}_1^Q)$, one deduces that the subalgebra $\mathscr{T}_G^*\mathbb{F}$ is involutive on $(G^n /\!\!/ G, \{\cdot\,,\cdot\}_n)$.

6.2. How many functions?

Beauville showed in 1990 that the natural functions $\mathrm{tr}X^k(a)$ on the loop algebra constitute an integrable system on a quotient of $\tilde{\mathfrak{g}}_n$ [5]. More precisely, here is his statement:

Let $\mathcal{V}_d := \left\{P(x, y) = y^N + s_1(x)y^{N-1} + \cdots + s_N(x) | \forall i, \deg s_i \le \mathrm{i}d\right\}$ and V_d be the set of elements P in $\mathcal{V}_d$ whose spectral curve $C_P := \{(x, y)|P(x, y) = 0\}$ is smooth. Let h_d be the map

$$\begin{aligned} h_d &: M_d \to V_d \\ X(\lambda) &\mapsto \det(y\mathrm{Id} - X(\lambda)) \end{aligned}$$

where M_d is the subset of $\tilde{\mathfrak{g}}_d$ of polynomial matrices whose characteristic polynomial is in V_d. This map is invariant under conjugation. Beauville denotes by Q_d the quotient M_d/G and $H_d : Q_d \to V_d$ the quotient map. He shows that the Poisson structures induced on Q_d by the classical linear Poisson structures give, for the functions $\mathrm{tr}\, X^k(a)$, the Hamiltonian vector fields

$$\mathcal{Y}_{k,a} : \dot{X}(\lambda) = c(a)\frac{\left[X(\lambda), X^k(a)\right]}{x - a},$$

where $c(a)$ is polynomial in a, depending on the chosen Poisson structure. He proves then the following theorem:

Theorem 25 *([5]). The Hamiltonian system $H_d : Q_d \to V_d$ is algebraic completely integrable with respect to the linear Poisson brackets.*

Indeed, he shows that the fiber of H_d over a generic point $P \in V_d$ is isomorphic to an affine open subset of the Jacobian of the spectral curve C_P. By dimensional consideration, he deduces, from this fact, that H_d defines $\frac{1}{2}N(dN + d + 2)$ independent functions on Q_d. Since these functions are in involution for the Poisson structure that he considers, the Hamiltonian system is Liouville integrable. In addition, the theorem claims that the Hamiltonian vector fields span, on the generic fiber, which is a complex torus, the space of linear vector fields.

Since we just want to count the number of independent functions defined by $\mathbb{F}$ on $\mathscr{A}/G$, we only need the independence of the functions on Q_d (with $d = n - 2$). Let us consider the following commutative diagram:

$$\begin{tikzcd} X(\lambda) \in \mathscr{A} \arrow[r, "h_n"] \arrow[d, "\alpha"] & \mathcal{V}_n \ni P(\lambda, y) \arrow[d, "\beta"] \\ \dfrac{X(\lambda) - (\lambda^n + 1)\,\mathrm{Id}}{\lambda} \in \tilde{\mathfrak{g}}_{n-2} \arrow[r, "h_{n-2}"] & \dfrac{1}{\lambda^N}\mathbb{C}[\lambda, y] \ni \dfrac{1}{\lambda^N}P(\lambda, \lambda y + \lambda^n + 1) \end{tikzcd}$$

In particular, $\beta(h_n(\mathscr{A})) \subset \mathcal{V}_{n-2}$. Moreover α is a G-equivariant diffeomorphism from $\mathscr{A}$ to $\tilde{\mathfrak{g}}_{n-2}$ so that we still have this diagram on the quotients with the maps H_n and H_{n-2}, replacing h_n and h_{n-2}. According to Beauville, the map H_{n-2} defines a family of $\frac{1}{2}N((n - 2)N + n)$ independent functions on $\tilde{\mathfrak{g}}_{n-2}/G$. Hence, H_n defines $s' \ge \frac{1}{2}N((n-2)N + n)$ independent functions on $\mathscr{A}/G$. Among this functions on $\mathscr{A}/G$, precisely $c' := Nn - 1$ are coming from det and are thus Casimir functions for our Poisson bracket $\{\cdot\,, \cdot\}_1^Q$ on $\mathscr{A}/G$.

Let us denote by m the dimension and by $2r$ the maximal rank of the Poisson manifold $\mathscr{A}/G$. Let $c = m - 2r$. Any involutive algebra of functions on $\mathscr{A}/G$ contains at most $r + c$ independent functions. Hence, $s' \le r + c$. On the other hand, $m = (n-2)N^2 + 1$ and we have $2(s' - c') + c' = N^2(n-2) + nN - nN + 1 = m$. Hence, $s' - c' = r$, $c' = c$ and H_n defines an integrable system on $\mathscr{A}/G$. This leads to the corollary:

Corollary 26. *The Hamiltonian system $H_n : \mathscr{A}/G \to V_n$ is Liouville integrable with respect to the Poisson structure $\{\cdot\,, \cdot\}_1^Q$.*

With some extra effort, one shows that it is actually also an algebraic completely integrable system, but we will not develop this here. In addition, we have the following proposition:

Proposition 27. *The transfer map $\mathscr{T}$ is a local diffeomorphism on an open dense subset of G^n.*

Proof. Since $\mathscr{T}$ is a polynomial map, we just have to prove that the differential of $\mathscr{T}$ is an isomorphism at a well-chosen point of G^n. Let $M = (M_1, \dots, M_n)$ be an n-tuple of diagonal matrices: $M_i = \mathrm{diag}(\lambda_{i,k}, 1 \le k \le N)$, for $1 \le i \le n$. Assume that these eigenvalues $\lambda_{i,k}$ are two by two distinct, and hence so are the polynomials $\lambda_{i,k}\lambda + 1$. Let Y be an element of the tangent space of G at M. Then the differential of $\mathscr{T}$ at M is written as

$$\mathsf{d}\mathscr{T}(M) \cdot H = \sum_{i=1}^{n} \overrightarrow{\prod_{j<i}}(\lambda M_j + \mathrm{Id})\lambda H_i \overrightarrow{\prod_{j>i}}(\lambda M_j + \mathrm{Id}),$$

where $\prod^{\rightarrow}$ is the ordered product. The coefficient (k, l), with $1 \le k, l \le N$, of this term is

$$(\mathsf{d}\mathscr{T}(M) \cdot H)_{kl} = \sum_{i=1}^{n} \prod_{j<i}(\lambda\lambda_{j,k} + 1)\lambda H_i^{kl} \prod_{j>i}(\lambda\lambda_{j,l} + 1).$$

Since the polynomials $\lambda_{i,k}\lambda + 1$ are mutually prime, a linear equality $(\mathsf{d}\mathscr{T}(M) \cdot H)_{kl} = 0$ leads to a polynomial equality in λ whose evaluations in the $-\lambda_{i,k}^{-1}$ and $-\lambda_{i,l}^{-1}$ yield $H_j^{kl} = 0$ for all $j \in [\![0, n]\!]$. Hence $\mathsf{d}\mathscr{T}(M) \cdot H = 0$ leads to $H = 0$. Thus $\mathsf{d}\mathscr{T}(M)$ is an isomorphism and, by analyticity, the transfer map $\mathscr{T}$ is a local diffeomorphism on an open dense subset of G^n. $\square$

Hence, we interpret the transfer map $\mathscr{T}$ as a morphism towards a well-known system on the loop algebra:

Theorem 28. *The Hamiltonian system $(G^n /\!\!/ G, \{\cdot\,,\cdot\}_n\,, \mathscr{T}^*\mathbb{F})$ is a Liouville integrable system. As a consequence, in view of Proposition 24 and Corollary 26, the transfer map*

$$\begin{aligned}\mathscr{T} : (G^n /\!\!/ G, \{\cdot\,,\cdot\}_n\,, \mathscr{T}^*\mathbb{F}) &\longrightarrow (\mathscr{A}/G, \{\cdot\,,\cdot\}_1^Q\,, \mathbb{F})\\ (M_1, \dots, M_n) &\longmapsto (\lambda M_1 + \mathrm{Id}) \dots (\lambda M_n + \mathrm{Id})\end{aligned}$$

is a morphism of integrable systems.

Proof. The two quotients $G^n /\!\!/ G$ and $\mathscr{A}/G$ have the same dimension, namely $(n-2)N^2+1$ and, since the transfer map is a local diffeomorphism, independence of involutive functions is preserved by the pull-back $\mathscr{T}^*$. Thus $\dim \mathscr{T}^*\mathbb{F} = \dim \mathbb{F}$. According to [19, Proposition 2.17], since $\mathscr{T}$ is a Poisson map, $\mathrm{Rk}(G^n /\!\!/ G) \ge \mathrm{Rk}(\mathscr{A}/G)$ and since, moreover, $\mathscr{T}$ is a local diffeomorphism, $\mathscr{T}^*\mathrm{Cas}(\mathscr{A}/G) \subset \mathrm{Cas}(G^n /\!\!/ G)$. It follows that $\mathrm{Rk}(G^n /\!\!/ G) = \mathrm{Rk}(\mathscr{A}/G)$ and the diffeomorphism $\mathscr{T}$ carries the integrable system from $\mathscr{A}/G$ to $G^n /\!\!/ G$. $\square$

7. The quadratic structure of the Toda lattice: A quasi-Poisson approach

In this last part, we propose a theorem generalizing the construction of a quasi-Poisson structure on the loop algebra (Section 4) to a more general Lie algebra $\mathfrak{g}$ of an associative algebra, equipped with a symmetric, non-degenerate, ad-invariant bilinear form $\langle\cdot|\cdot\rangle$, which satisfies, for all $x, y, z \in \mathfrak{g}$, $\langle xy|z\rangle = \langle x|yz\rangle$. Our precise assumption, through this section, is that $\mathfrak{g}$ admits a Lie algebra splitting as a vector space direct sum of three Lie algebras of associative subalgebras:

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-,$$

where $\mathfrak{g}_0$ is finite-dimensional and such that, with respect to $\langle\cdot|\cdot\rangle$, $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are two dual spaces and $\mathfrak{g}_0^* = \mathfrak{g}_0$. Let P_+, P_- and P_0 be the linear projectors on $\mathfrak{g}_+, \mathfrak{g}_-$ and $\mathfrak{g}_0$ respectively and $R := P_+ + P_0 - P_-$. Let G_0 be a Lie group whose Lie algebra is $\mathfrak{g}_0$. Let $\mathcal{F}(\mathfrak{g})$ be the algebra of functions on $\mathfrak{g}$ which are polynomial in the linear maps $x \mapsto \langle x|y\rangle$, $y \in \mathfrak{g}$. On this algebra, the gradient is well defined:

$$\forall y \in \mathfrak{g}, \quad \langle \nabla f(x)|y\rangle = \frac{\mathrm{d}}{\mathrm{d}t}_{|_{t=0}} f(x + ty).$$

Theorem 29. *The bivector field $\{\cdot\,,\cdot\}_R^Q$, defined on $\mathcal{F}(\mathfrak{g})$ by, $\forall f_1, f_2 \in \mathcal{F}(\mathfrak{g})$, $\forall x \in \mathfrak{g}$,*

$$\{f_1, f_2\}_R^Q(x) := \frac{1}{2}(\langle [x, \nabla f_1(x)]|R(x\nabla f_2(x) + \nabla f_2(x)x)\rangle - \langle [x, \nabla f_2(x)]|R(x\nabla f_1(x) + \nabla f_1(x)x)\rangle)$$

is a quasi-Poisson structure on the G_0-manifold $\mathfrak{g}$. For this structure $\mathrm{ad}_{\mathfrak{g}}$-invariant functions are in involution and the associated Hamiltonian vector fields commute.

Proof. The proof is the same as in the loop algebra case, computing the Jacobiator of $\{\cdot,\cdot\}_R^Q$ or using the tensor form of R and its skew-symmetric part in $\mathcal{T}_2(\mathfrak{g}) := \left\{\sum_{i,j} c_{i,j} e_i \otimes e_j | c_{i,j} \in \mathbb{C}\right\}$, where $(e_i)_{i\in I}$ is a basis of $\mathfrak{g}$. □

According to the theorem, we obtain a family of functions on $\mathfrak{g}$, which are in involution with respect to the quasi-Poisson structure. In combination with the following proposition, this leads in many cases to enough independent involutive functions on a quotient to insure integrability.

Proposition 30. *The subspaces $\mathfrak{g}_0 \oplus \mathfrak{g}_+$ and $\mathfrak{g}_+$ are two quasi-Poisson G_0-submanifolds (i.e., the inclusions are quasi-Poisson maps).*

Proof. Let $f_1 \in \mathcal{F}(\mathfrak{g})$ be null on the subspace $\mathfrak{g}_0 \oplus \mathfrak{g}_+$ and $x \in \mathfrak{g}_0 \oplus \mathfrak{g}_+$. Since $\mathfrak{g}_\pm^* = \mathfrak{g}_\mp$ and $\mathfrak{g}_0^* = \mathfrak{g}_0$, we have $\nabla f_1(x) \in \mathfrak{g}_+$. Moreover these assumptions also imply that $\mathfrak{g}_+\mathfrak{g}_0 \subset \mathfrak{g}_+$, so that $\nabla f_1(x)x \in \mathfrak{g}_+$ and $R(\nabla f_1(x)x) = \nabla f_1(x)x$ (the same holds for R^* and with $x\nabla f_1(x)$). Thus we have, $\forall f_2 \in \mathcal{F}(\mathfrak{g})$,

$$\begin{aligned}\{f_1, f_2\}_R^Q(x) &= \frac{1}{2}(\langle -[x, \nabla f_1(x)] | x\nabla f_2(x) + \nabla f_2(x)x\rangle - \langle [x, \nabla f_2(x)] | x\nabla f_1(x) + \nabla f_1(x)x\rangle)\\ &= -\langle x\nabla f_1(x) | x\nabla f_2(x)\rangle + \langle \nabla f_1(x)x | \nabla f_2(x)x\rangle\\ &= 0.\end{aligned}$$

The same argument holds for the case of $\mathfrak{g}_+$. □

As an illustration, let us apply the Theorem 29 within the framework of the symmetric Toda matrix. The space which interests us is the set $\mathscr{M}$ of symmetric matrices $A = [A_{i,j}]_{i,j}$ such that $\forall i, j$, if $|i-j| > 1$ then $A_{i,j} = 0$:

$$\mathscr{M} := \left\{ \left. \begin{pmatrix} b_1 & a_1 & 0 & \dots & \dots & 0 \\ a_1 & b_2 & a_2 & & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \dots & & \dots & a_{N-1} & b_N \end{pmatrix} \right| a_i, b_j \in \mathbb{C} \right\}$$

In order to define a quadratic bivector field on this space, we first consider the Lie algebra splitting of the Lie algebra $\mathfrak{g}$ of the associative algebra $\mathfrak{gl}(N,\mathbb{C})$:

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{d} \oplus \mathfrak{l},$$

where $\mathfrak{u}$ (resp. $\mathfrak{l}$) is the subspace of strictly upper (resp. strictly lower) triangular matrices and $\mathfrak{d}$ is the subspace of diagonal matrices. When $\mathfrak{g}$ is equipped with the bilinear form $\langle A|B\rangle = \mathrm{tr}(AB)$ the assumptions of Theorem 29 are satisfied. Thus, writing $R = P_{\mathfrak{u}} + P_{\mathfrak{d}} - P_{\mathfrak{l}}$, the formula

$$\{f_1, f_2\}_R^Q(x) := \frac{1}{2}(\langle [x, \nabla f_1(x)] | R(x\nabla f_2(x) + \nabla f_2(x)x)\rangle - \langle [x, \nabla f_2(x)] | R(x\nabla f_1(x) + \nabla f_1(x)x)\rangle)$$

defines a quasi-Poisson structure on the D-manifold $\mathfrak{g}$, where D is the group of invertible diagonal $N\times N$ matrices and acts on $\mathfrak{g}$ by conjugation. Since this group is commutative, the Cartan 3-form $\phi_{\mathfrak{d}}$ vanishes and the quasi-Poisson bracket $\{\cdot,\cdot\}_R^Q$ is in fact a Poisson bracket. If $(x_{ij})_{1\le i,j\le N}$ denote the coordinate functions on $\mathfrak{g}$, one has, $\forall i, j, k, l \in [\![1, N]\!]$,

$$\left\{x_{ij}, x_{kl}\right\}_R^Q = (\varepsilon_{jl} + \varepsilon_{ik}) x_{kj} x_{il} + (\delta_{il} - \delta_{jk}) x_{ij} x_{kl}$$

where $\varepsilon_{ij} = 1$ if $i > j$, 0 if $i = j$ and -1 if $i < j$. These formulas allow us to show very simply that the subspace $\mathscr{D}_3$ of tridiagonal matrices is a Poisson submanifold of $\mathfrak{g}$, just computing the brackets $\left\{x_{ij}, x_{kl}\right\}_R^Q$ where $|i-j| \ge 2$. On the other hand $\mathscr{M}$ is not a Poisson submanifold of $\mathfrak{g}$. However, the following lemma due to Fernandes and Vanhaecke authorizes us to equip $\mathscr{M}$ with an inherited Poisson structure.

Lemma 31 ([7]). *Let $(M, \{\cdot\,,\cdot\})$ be a Poisson manifold equipped with an involution σ which is a Poisson map. Let N be the submanifold of M consisting of the fixed points of σ. Denote by ι the inclusion map $\iota : M \hookrightarrow N$. Then N carries a unique Poisson structure $\{\cdot\,,\cdot\}_N$ such that*

$$\iota^*\{f_1, f_2\} = \left\{\iota^* f_1, \iota^* f_2\right\}_N$$

for all $f_1, f_2 \in \mathcal{F}(M)$ that are σ-invariant.

In our case, transposition is a Poisson involution admitting $\mathscr{M}$ as a set of fixed points. The inherited structure is given by

$$\{a_i, a_{i-1}\}_R^Q = -\frac{1}{2}a_i a_{i-1} \qquad \{a_i, b_i\}_R^Q = -a_i b_i \qquad \{b_i, b_{i-1}\}_R^Q = -2a_{i-1}^2$$
$$\{a_i, a_{i+1}\}_R^Q = \frac{1}{2}a_i a_{i+1} \qquad \{a_i, b_{i+1}\}_R^Q = a_i b_{i+1} \qquad \{b_i, b_{i+1}\}_R^Q = 2a_i^2$$

(all other brackets being null). We obtain, using our formalism, the same quadratic Poisson bracket for the Toda lattice as Damianou in [6].

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